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RELATIVE CAUCHY EVOLUTION
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Abstract (Italiano)

Nel corso degli anni '60 del secolo scorso è iniziata la ricerca di una formulazione matematicamente rigorosa della teoria quantistica dei campi. Uno dei primi rilevanti successi in questo ambito si deve all'approccio algebrico e assiomatico proposto da Haag e Kastler (si veda [HK64]). Tale formalismo consente di definire la teoria quantistica dei campi sullo spaziotempo di Minkowski in un ben preciso contesto matematico, quello algebrico, e di implementare in maniera naturale all'interno di questa teoria i concetti di causalità e di covarianza di Lorentz.

Precisiamo che questo tipo di approccio non genera una teoria nuova rispetto alla teoria quantistica dei campi sullo spaziotempo di Minkowski nella sua formulazione originaria. Al contrario riproduce i medesimi risultati, presentando tuttavia due vantaggi significativi: in primo luogo la formulazione della teoria avviene in un contesto matematico ben precisato, che consente di motivare in maniera rigorosa i risultati ottenuti, e in secondo luogo l'approccio si rivela adatto a notevoli estensioni. Infatti nel corso degli anni le idee originali di Haag e Kastler si sono sviluppate e hanno visto ampliare il proprio dominio di applicazione, pur conservando in buona parte la loro identità, sino a giungere alla formulazione della teoria quantistica dei campi su spazitempi curvi.

A quasi 40 anni di distanza dal lavoro di Haag e Kastler, Brunetti, Fredenhagen e Verch ([BFV03]) hanno proposto un approccio alla teoria dei campi su spazitempi curvi che va sotto il nome di *principio di località generalmente covariante*. Questo approccio è da considerarsi come complementare a quello originale in quanto non introduce nuovi assiomi nella teoria e consente di recuperare in maniera naturale l'approccio algebrico e assiomatico di Haag e Kastler. D'altra parte ha il merito di porre l'accento sugli aspetti che accomunano le procedure di quantizzazione su spazitempi distinti (ovvero la struttura funtoriale soggiacente) e sulle caratteristiche che invece le contraddistinguono (ovvero gli spazi di stati). Inoltre la struttura funtoriale di questo approccio implementa naturalmente la proprietà di covarianza nella teoria quantistica di campo, così come è previsto dalla relatività generale per ogni teoria fisica.

Nella tesi è presentato in un contesto generale il *principio di località generalmente covariante*. Questo postula che ogni teoria quantistica di campo sia formulata come una *teoria quantistica di campo localmente covariante* (nel seguito talvolta abbre-

viata dall'acronimo *LCQFT*). Senza la pretesa di essere esaustivi, possiamo dire che una *LCQFT* consiste in un funtore covariante che a ogni spaziotempo globalmente iperbolico associa un'algebra e a ogni embedding isometrico tra spazitempi globalmente iperbolici fa corrispondere un omomorfismo iniettivo tra le algebre associate a tali spazitempi. Due ulteriori proprietà possono essere richieste a una *LCQFT*: la *causalità*, ovvero, semplificando, il fatto che commutino tra loro gli elementi di due algebre associate a spazitempi che ammettono embedding isometrici con immagini causalmente separate in uno spaziotempo comune, e il *time slice axiom*, ossia la richiesta che sia suriettivo ogni omomorfismo associato a un embedding isometrico la cui immagine contiene una superficie di Cauchy del suo codominio.

Ribadiamo che la covarianza generale è implementata all'interno della teoria grazie alla proprietà di covarianza del funtore che realizza certa *teoria quantistica di campo localmente covariante*. Sulla scia di quanto provato da Brunetti, Fredenhagen e Verch, riproponiamo la dimostrazione del fatto che da ogni *LCQFT causale* verificante il *time slice axiom* è possibile recuperare lo schema assiomatico di Haag e Kastler, il quale coinvolge reti di algebre locali e automorfismi covarianti associati alle isometrie dello spaziotempo soggiacente. Questo fatto consente di interpretare una opportuna sottoalgebra dell'algebra associata da una *LCQFT* ad un dato spaziotempo come l'algebra delle osservabili fisiche associate a tale spaziotempo.

L'approccio alla teoria quantistica di campo suggerito dal *principio di località generalmente covariante* è completamente indipendente dal particolare modello fisico che di volta in volta può essere preso in considerazione, tuttavia, affinché il principio si dimostri fisicamente rilevante, occorre verificare la possibilità di realizzare una *teoria quantistica di campo localmente covariante* che soddisfi sia la *causalità* che il *time slice axiom* in tutte le situazioni di interesse fisico. Nella tesi si riprendono i risultati ottenuti in [BFV03] per il campo di Klein-Gordon e si discutono i casi del campo di Proca e del campo elettromagnetico. Cogliamo l'occasione per ricordare che il caso del campo di Dirac è stato affrontato in [San10b].

Come si vedrà, di fatto la realizzazione di una *teoria quantistica di campo localmente covariante* per un campo bosonico riposa soltanto sulla possibilità di costruire uno spazio simplettico di soluzioni per le equazioni di campo classiche per ogni spaziotempo globalmente iperbolico e sulla individuazione di una mappa simplettica in corrispondenza di ogni embedding isometrico tra spazitempi globalmente iperbolici, mappa simplettica che ha come dominio e codominio gli spazi simplettici associati agli spazitempi che fanno da dominio e da codominio per l'embedding assegnato. Per quanto riguarda il soddisfacimento della *causalità* e del *time slice axiom* di una *LCQFT* ottenuta in questo modo, di nuovo il problema si riduce a livello classico a questioni di supporto delle soluzioni di problemi di Cauchy per le equazioni di campo e alla suriettività della mappe simplettiche.

Obiettivo principale di questa tesi è lo studio di un particolare tipo di dinamica

introdotto in [BFV03] che va sotto il nome di *evoluzione relativa di Cauchy* (*RCE*). La caratteristica peculiare della *RCE* risiede nella sua capacità di evidenziare la sensibilità di una *teoria quantistica di campo localmente covariante* alle fluttuazioni della metrica dello spaziotempo sottostante. Precisamente ci si pone lo scopo di studiare la relazione che intercorre tra la *RCE* e il tensore energia-impulso nel caso delle *LCQFT* costruite per il campo di Klein-Gordon, per il campo di Proca e per il campo elettromagnetico. L'interesse nei confronti di tale relazione nasce dall'intento di incorporare il valore di aspettazione del tensore energia-impulso di un campo quantistico assegnato nel membro di destra dell'equazione di Einstein (per maggiori dettagli sull'equazione di Einstein semiclassica rimandiamo a [Wal94]).

Seguendo la definizione proposta recentemente da Fewster e Verch in [FV11], limitatamente a quelle *teorie quantistiche di campo localmente covarianti* che soddisfano il *time slice axiom*, definiamo l'*evoluzione relativa di Cauchy* come un automorfismo sull'algebra associata a un dato spaziotempo globalmente iperbolico indotto da una perturbazione locale della metrica spaziotemporale. La definizione stessa della *RCE* consente di interpretarla come una sorta di reazione dinamica della teoria quantistica di campo a una fluttuazione della metrica dello spaziotempo sottostante. Riesamineremo alcune proprietà della *RCE* ponendo l'accento sulla sua insensibilità a perturbazioni della metrica indotte da diffeomorfismi e sul fatto che, di conseguenza, la derivata funzionale della *RCE* rispetto alla metrica abbia divergenza nulla.

In [BFV03] è sviluppato nel dettaglio lo studio dell'*evoluzione relativa di Cauchy* per il campo di Klein-Gordon. In particolare Brunetti, Fredenhagen e Verch giungono a dimostrare una particolare relazione che in questa situazione intercorre tra *RCE* e tensore energia-impulso. Qui questo caso è riesaminato a scopo esemplificativo e ci si pone l'obiettivo di estendere la relazione tra *RCE* e tensore energia-impulso dimostrata in [BFV03] per il campo di Klein-Gordon anche ai casi del campo di Proca e del campo elettromagnetico (per l'analogo problema nel caso del campo di Dirac si rimanda di nuovo a [San10b]). In questo modo il significato dell'*evoluzione relativa di Cauchy* in relazione al tensore energia-impulso risulta esteso dal caso del campo di Klein-Gordon ai casi del campo di Proca e del campo elettromagnetico. In particolare questo fatto motiva l'introduzione del valore di aspettazione del tensore energia-impulso nel membro di destra dell'equazione di Einstein anche per i casi del campo di Proca e del campo elettromagnetico.

Abstract (English)

During the Sixties of the last century the search for a mathematically rigorous formulation of quantum field theory has begun. One of the first and most prominent successes in this area is due to the algebraic and axiomatic approach proposed by Haag and Kastler (refer to [HK64]). This formalism allows the definition of quantum field theory over Minkowski spacetime in a precisely specified mathematical context, namely the algebraic one, and the natural implementation of the notions of causality and Lorentz covariance in such theory.

We specify that this approach does not produce a new theory with respect to the original formulation of quantum field theory on Minkowski spacetime. On the contrary it gives rise to equivalent results, yet presenting two significant advantages: in first place the theory is formulated in a precise mathematical context, that allows to motivate rigorously the results one obtains, and in second place the approach proves suitable to remarkable extensions. As a matter of fact over the years the original ideas of Haag and Kastler were significantly developed and went through an enlargement of their range of applicability, while largely preserving their original identity, until the formulation of quantum field theory on curved spacetimes.

Almost 40 years after the work made by Haag and Kastler, Brunetti, Fredenhagen and Verch ([BFV03]) proposed a new approach to quantum field theories on curved spacetimes named *generally covariant locality principle*. On one hand this approach is to be considered as complementary to the original one since it does not add new axioms to the theory and allows the natural recovering of the algebraic and axiomatic approach by Haag and Kastler. On the other hand it has the merit of highlighting the common aspects of quantization procedures on different spacetimes (namely the underlying functorial structure) and the distinguishing features (namely state spaces). Furthermore the functorial structure of this approach naturally implements covariance in quantum field theories, as it is expected by each physical theory according to general relativity.

In this thesis the *generally covariant locality principle* is presented in a general setting. It postulates that each quantum field theory be formulated as a *locally covariant quantum field theory* (sometimes denoted by the acronym *LCQFT*). Without pretending to be exhaustive, we may say that a *LCQFT* consists of a covariant functor mapping each globally hyperbolic spacetime to an algebra and each isometric em-

bedding between two globally hyperbolic spacetimes to an injective homomorphism between the algebras associated to such spacetimes. Other two properties can be required to a *LCQFT*: *causality*, which, simplifying, means that elements coming from two algebras associated to spacetimes isometrically embedded in causally separated subregions of a common spacetime commute, and the *time slice axiom*, which requires that each homomorphism associated to an isometric embedding whose image includes a Cauchy surface of its codomain be surjective.

We repeat that general covariance is implemented in the theory as a consequence of the covariance property of the functor giving rise to a *locally covariant quantum field theory*. Following what was shown by Brunetti, Fredenhagen and Verch, we present the proof of the fact that, starting from a *LCQFT* fulfilling both *causality* and the *time slice axiom*, it is possible to recover the Haag-Kastler scheme, involving nets of local algebras and covariant automorphisms associated to isometries of the underlying spacetime. This fact makes it possible to interpret a proper subalgebra of the algebra provided by a *LCQFT* on a given spacetime as the algebra of physical observables associated to that spacetime.

The approach to quantum field theory suggested by the *generally covariant locality principle* is completely independent of the specific physical model considered from time to time, yet we must check the possibility of realizing a *locally covariant quantum field theory* fulfilling both *causality* and the *time slice axiom* in each situation of physical interest in order to have a physically relevant principle. In this thesis the results obtained in [BFV03] for the Klein-Gordon field are recovered and the cases of the Proca and the electromagnetic fields are discussed. We take the chance to remind that the case of the Dirac field was handled in [San10b].

As we will see, the construction of a *locally quantum field theory* for a bosonic field actually relies only on the possibility of building a symplectic space of solutions for the classical field equations for each globally hyperbolic spacetime and on the specification of a symplectic map for each isometric embedding between two globally hyperbolic spacetimes, the domain and codomain of the symplectic map being the symplectic spaces associated to the domain and codomain of the given embedding. As for the *causality property* and the *time slice axiom* of a *LCQFT* built in this way, again the problem is reduced at a classical level to a matter of support for solutions of Cauchy problems for the field equations and to the surjectivity of the symplectic maps.

The main purpose of this thesis is to study a particular type of dynamics proposed by [BFV03] named *relative Cauchy evolution* (briefly *RCE*). The distinctive feature of the *RCE* relies in its ability of highlighting the sensitivity of a *locally covariant quantum field theory* to fluctuations of the metric of the underlying spacetime. In particular our aim is to study a relation between the *RCE* and the stress-energy tensor for the *LCQFTs* built for the Klein-Gordon field, the Proca field and the

electromagnetic field. The interest in such relation arises from the intention of including the expectation value of the stress-energy tensor of a given quantum field in the right hand side of the Einstein's equation (for further details on the semiclassical Einstein's equation we refer to [Wal94]).

Following the definition recently proposed by Fewster and Verch in [FV11], only for those *locally covariant quantum field theories* fulfilling the *time slice axiom*, we define the *relative Cauchy evolution* as an automorphism on the algebra associated to a given globally hyperbolic spacetime induced by a local perturbation of the spacetime metric. The definition of the *RCE* suggests its interpretation as a dynamical reaction of the quantum field theory to a fluctuation of the metric of the underlying spacetime. We will re-examine some properties of the RCE with particular attention to its insensitivity to perturbations of the metric induced by diffeomorphisms and to the fact that, consequently, the functional derivative of the RCE with respect to the spacetime metric has null divergence.

In [BFV03] the relative Cauchy evolution for the Klein-Gordon field is thoroughly analyzed. In particular Brunetti, Fredenhagen and Verch were successful in showing that in this case a particular relation between the RCE and the stress-energy tensor holds. Here we re-examine this case as an example and we have as our goal to extend to the cases of the Proca and the electromagnetic fields the relation between the RCE and the stress-energy tensor proved in [BFV03] for the Klein-Gordon field (for the similar problem in the case of the Dirac field we refer again to [San10b]). In this way the meaning of the RCE in relation to the stress-energy tensor is extended from the case of the Klein-Gordon field to the Proca and the electromagnetic fields. In particular this fact motivates the insertion of the expectation value of the stress-energy tensor on the right hand side of the Einstein equation for the Proca and the electromagnetic fields too.

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Introduction

In the mid Sixties Haag and Kastler proposed an algebraic approach to quantum field theory on Minkowski spacetime ([HK64]). Although it is equivalent to the original formulation of quantum field theory arising from the Wightman axioms ([SW64]), this approach proved to be very successful since it provided a mathematically precise framework for quantum field theories which could be easily applied on curved spacetimes.

In this context a further milestone ahead was unveiled by Brunetti, Fredenhagen and Verch in [BFV03]. To wit they formulated the *generally covariant locality principle* (in the following denoted by *GCLP*), postulating that each quantum field theory on an arbitrary globally hyperbolic spacetime must be provided by a *locally covariant quantum field theory* (*LCQFT*), i.e. a covariant functor from the category of globally hyperbolic spacetimes to the category of algebras. The result is a formulation of quantum field theory that naturally exhibits the covariance property required by general relativity, this being a direct consequence of the functorial structure of each LCQFT.

As suggested in [BFV03], one can require two additional properties to a LCQFT:

- *causality*, which, roughly speaking, means that we require that elements of the algebras, which are associated via a fixed LCQFT to globally hyperbolic spacetimes embedded in causally separated subregions of another globally hyperbolic spacetime, must commute;
- the *time slice axiom*, which requires that each morphism of the category of algebras must be surjective if it is obtained applying a given LCQFT to a morphism of the category of globally hyperbolic spacetimes, whose image includes a Cauchy surface of the target spacetime.

Causality forces the absence of causal relations between observables localized in causally separated subregions of a globally hyperbolic spacetime. This simply means that we do not admit causal effects between events not connected by causal curves. As for the time slice axiom, we can interpret it as a sort of causal determinacy, in analogy with the classical case. As much as we know everything about a classical dynamical system once suitable initial data on a Cauchy surface of a globally hyperbolic spacetime are assigned, likewise the whole algebra of observables associated to

a quantum field on a globally hyperbolic spacetime is contained in the algebra of observables associated to a suitable neighbourhood of a Cauchy surface.

In [BFV03] it was shown that, on each globally hyperbolic spacetime, an arbitrary causal LCQFT automatically gives rise to a quantum field theory satisfying the Haag-Kastler axioms ([HK64]). Hence we may regard the GCLP as a natural criterion to realize on curved spacetimes the approach to quantum field theory originally proposed by Haag and Kastler. Moreover we may borrow the interpretation of the Haag-Kastler axioms saying that a proper subalgebra of the algebra assigned by a fixed LCQFT applied to any but fixed globally hyperbolic spacetime is the algebra of the quantum observables admitted by the physics on the given spacetime.

The GCLP proved to be very successful. A number of results in various topics about quantum field theories on curved spacetimes were proved in this framework. For example LCQFTs fulfilling both causality and the time slice axiom were built for free field models of physical interest (namely Klein-Gordon, Dirac, Proca and electromagnetic fields) and questions about what it is meant for a theory to produce the same physics in all spacetimes arose. A few references are [BFV03, BGP07, San10b, Dap11, FV11].

Indeed this is not the whole story for quantum field theories on curved spacetimes. In fact at this point we are not able to get physical predictions from the algebra of observables. What we need is a notion of state to be evaluated on the observables in order to get predictions exactly as we do in quantum mechanics. This issue is not touched by the GCLP, nor we discuss it in this thesis. Yet we feel worth to say that relevant results were obtained also in this sector. For example it is known that there exist states for quantum field theories on globally hyperbolic spacetimes which satisfy properties that are known to hold for the vacuum states of quantum field theories on Minkowski spacetime (e.g. the Hadamard condition and the Reeh-Schlieder property). Some references for these topics are [Kay91, Rad96, SV01, SVW02, FV03, FP03, San10a, Dap11].

Another interesting application of the GCLP consists in the realization of a particular form of dynamics known as *relative Cauchy evolution (RCE)*. The RCE is an algebraic automorphism that can be defined on each globally hyperbolic spacetime and for each LCQFT fulfilling the time slice axiom. Its relevance relies in the fact that it accounts for the effects that a fluctuation of the spacetime metric produces on the algebra provided by the LCQFT on a given globally hyperbolic spacetime.

The study of the RCE is interesting in first place because indeed we want to deal with a stable theory, which is to say that it would be unlikely to have a quantum field theory on a globally hyperbolic spacetime with observables that are so much sensitive to small changes in the spacetime metric that they disappear (or maybe appear) only because of a small change in the spacetime geometry. In the second place the interest in the analysis of the reaction of a quantum field theory to fluctuations of

the spacetime metric comes from the attempt to solve the semiclassical Einstein's equation (we only give a sketch of the problem). Up to now our quantum field theories (and this is the case of the GCLP too) are settled on spacetimes which are given once and for all. Yet, as far as we know, the spacetime where we live is a solution of the Einstein's equation. To simplify the situation assume that in the whole universe there is nothing but a quantum field. Then one should insert the expectation value of the stress-energy tensor associated to such field on the RHS of the Einstein's equation (the equation that arises is the above mentioned semiclassical Einstein's equation, see [Wal94] for further reference). When one tries to solve the semiclassical Einstein's equation, serious difficulties emerge: As the solution develops, the quantum field given at the beginning is affected by the new geometry of the spacetime where it lives. Hence we have a back-reaction effect, namely the quantum field, whose stress-energy tensor appears on the RHS of the semiclassical Einstein's equation, is affected by the solution of such equation. If the quantum field theory is too much sensitive to a change in the spacetime structure (essentially a change in the metric), it may happen that the stress-energy tensor appearing on the RHS of the semiclassical Einstein's equation loses its meaning while we solve the equation (as a matter of fact it happens that we no longer have any equation to solve). Being able to properly control the RCE means that the algebra of observables provided by a given LCQFT on some globally hyperbolic spacetime is not severely distorted by a small change in the spacetime metric, hence we can expect that the stress-energy tensor associated to the quantum field preserves its meaning while we solve the semiclassical Einstein's equation, i.e. it still describes the stress-energy tensor associated to the quantum field taken into account even when the spacetime geometry has changed due to the fact that we are solving the Einstein's equation.

Now that we have given a sketch of the topics we are going to deal with and we have presented the motivation that pushed us to their study, we would like to briefly summarize the content of the thesis.

In Chapter 1 we present almost all the mathematical tools that will be needed for the next chapters. We devote Section 1.1 to introduce some notions in differential geometry, namely manifolds and vector bundles. Particular attention is devoted to differential forms and integration over manifolds. In Section 1.2 we specialize to the case of Lorentzian manifolds, being interested in the notion of global hyperbolicity. With these concepts at hand, in Section 1.3 we turn our attention to the discussion of wave equations on globally hyperbolic spacetime. In first place we define what we mean by wave equation (or normally hyperbolic equation to be more precise) and in second place we present a theorem about the existence and uniqueness of solutions for Cauchy problems associated to normally hyperbolic equations, we introduce Green operators and we study some of their properties. In Section 1.4 we completely change

the subject in order to deal with algebras and states. We are mainly interested in unital C^* -algebras (in particular Weyl systems and CCR representations, which are special C^* -algebras that bestly fit the canonical commutation relations) and states defined on them. We conclude the first chapter with Section 1.5, where we recall some basic concepts from category theory.

The main discussion begins with Chapter 2. In Section 2.1 the generally covariant locality principle (GCLP) is formulated defining the notion of locally covariant quantum field theory (LCQFT) and a physical interpretation of the principle is provided, interpretation that is essentially borrowed from that of the Haag-Kastler axioms (refer to [HK64]). We conclude this section showing that it is possible to rigorously recover the Haag-Kastler axioms (hence their interpretation) once that a LCQFT fulfilling the causality condition and the time slice axiom is given. We devote Section 2.2 to show a procedure to build a LCQFT starting from the assignment of a proper normally hyperbolic equation involving sections in a general vector bundle over a globally hyperbolic spacetime. Such procedure essentially consists in the construction of a covariant functor describing the theory of the classical field and in the quantization of this theory via composition with a properly defined covariant functor which embodies the quantization scheme. Section 2.3 concludes the second chapter presenting the realization of LCQFTs for three models of physical interest, namely the Klein-Gordon field, the Proca field and the electromagnetic field. While the Klein-Gordon field is a mere specialization of the general procedure presented in Section 2.2, the other two require significant modifications due to the fact that their classical dynamics is not ruled by a normally hyperbolic equation.

We conclude the thesis with Chapter 3 discussing the relative Cauchy evolution (RCE). In Section 3.1 we define the RCE for a LCQFT fulfilling the time slice axiom and we study its insensitivity to fluctuations of the perturbed spacetime metric produced by diffeomorphisms. After that we introduce the functional derivative of the RCE with respect to the spacetime metric as a section in the symmetrized tensor product of two copies of the tangent bundle and we show that its divergence (with respect to the Levi-Civita connection) is null. These properties, namely symmetry and null divergence, are hints for a strict relation between the functional derivative of the RCE and the stress-energy tensor associated to some quantum field. The study of this relation for the specific cases of the Klein-Gordon, the Proca and the electromagnetic fields concludes the thesis. Specifically in Section 3.2, after a brief summary of some of the properties satisfied by quasifree Hadamard states, we present the calculation originally performed in [BFV03] to prove that a strict relation between the RCE and the quantized stress-energy tensor holds for the Klein-Gordon field and we show that an identical relation holds for the Proca and the electromagnetic fields too.

Chapter 1

Mathematical preliminaries

We devote the present chapter to the introduction of the main mathematical tools which will be indispensable for the discussion in the following chapters. All the topics presented here are discussed very briefly and the interested reader is invited to refer to the specific literature of each sector. For this scope at the beginning of all sections we provide some reference for the subject discussed.

The first section is devoted to the definition of manifolds, vector bundles and connections, differential forms and integration. In the second section we present few arguments concerning Lorentzian geometry. Then the third section is devoted to some basic topics about wave equations on globally hyperbolic spacetimes: we present a theorem about existence and uniqueness of solutions to such equations with proper initial data and then we will introduce the advanced and retarded Green operators together with their properties. In the fourth section of this chapter we turn our attention to the mathematical ingredients that will be essential in the construction of the algebraic approach to quantum field theory, specifically C^* -algebras and states. Finally the last section presents some very useful concepts of category theory that will be widely applied in the next chapters.

1.1 Differential geometry

This section is a very concise (and far from complete) recollection of the notions in differential geometry that are unavoidable for our discussion. Besides the efforts spent in making this section self sufficient, almost all topics are presented in a manner that is too brief to be clear for a reader that approaches to them for the first time. For this reason the author strongly encourages the reader to refer to any book concerning differential geometry (for example [\[Jos95\]](#) or [\[Boo86\]](#)) to clarify the omissions to which we are forced.

1.1.1 Manifolds and tensor bundles

We begin defining manifolds. These objects will provide the playground for the entire thesis. The notion of manifold that we present is not the more general one. To be precise we define smooth connected Hausdorff manifolds with a countable basis of open subsets. This is a sufficiently wide class of manifolds and at the same time it incorporates a number of properties we are interested in.

Definition 1.1.1. A d -dimensional manifold M is a connected Hausdorff topological space with a countable basis of open subsets such that for each point $p \in M$ there exists a triple (U, Ω, ϕ) , called *coordinate neighborhood* (or *local chart*), where U is an open neighborhood of p in M , Ω is an open neighborhood of 0 in \mathbb{R}^d and $\phi : U \rightarrow \Omega$ is a homeomorphism. There are two other requirements:

- there exists a (*smooth*) *atlas*, which is a collection $\{(U_\alpha, \Omega_\alpha, \phi_\alpha)\}_{\alpha \in I}$ of coordinate neighborhoods in M , where I is an index set, such that $\{U_\alpha\}_{\alpha \in I}$ is an open covering of M and the map, called *transition chart*,

$$\begin{aligned} T_{\phi_\alpha}^{\phi_\beta} : \Omega_\alpha \cap \Omega_\beta &\rightarrow \Omega_\alpha \cap \Omega_\beta \\ x &\mapsto (\phi_\beta \circ \phi_\alpha^{-1})(x) \end{aligned}$$

is a diffeomorphism for each $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$;

- there exists a *maximal atlas*, i.e. an atlas that contains each coordinate neighborhood (U, Ω, ϕ) such that the transition maps $T_{\phi_\alpha}^\phi$ and $T_\phi^{\phi_\alpha}$ are diffeomorphisms for each $\alpha \in I$ with $U_\alpha \cap U \neq \emptyset$.

We would like to make some remarks concerning this definition. In the first place each atlas of a manifold is contained in a maximal one, so that it is sufficient to find an atlas and then the maximal atlas is automatically obtained. This implies that a connected Hausdorff space with a countable basis becomes a manifold if it possesses an atlas, even if not maximal. Secondly we observe that the topology of each manifold defined here is such that it is also a paracompact space and this implies that for each of our manifolds there exists a partition of unity (cfr. [Boo86, Chap. V, Sect. 4, p. 193]).

Now that we have a notion of manifold, we would like to define “regular” functions between manifolds (continuous functions are already defined since manifolds are topological spaces).

Definition 1.1.2. Let M and N be two manifolds and let f be a continuous function from M to N . We say that f is a C^k -function if for each $p \in M$, each coordinate neighborhood (U, Ω, ϕ) of p in M and each coordinate neighborhood (V, Θ, ψ) of

$f(p)$ in N , the function

$$\begin{aligned} f_{U,V} : \phi(U \cap f^{-1}(V)) &\rightarrow \psi(f(U) \cap V) \\ x &\mapsto (\psi \circ f \circ \phi^{-1})(x) \end{aligned}$$

is of class C^k (in the sense of functions between open subsets of Euclidean spaces).

Moreover f is a *smooth function* if it is a C^k -function for each $k \in \mathbb{N}$ and we say that f is a *diffeomorphism* if it is a homeomorphism which is smooth together with its inverse.

Given a manifold M and a notion of smooth function, for each $p \in M$ it is possible to introduce a vector space $T_p M$, called tangent space that proves very useful when one wants to speak of “derivatives” at the point p of real valued functions defined on M .

Definition 1.1.3. Let M be a d -dimensional manifold and let $p \in M$. Consider the set \mathcal{C}_p of smooth curves $c : I \rightarrow M$, where I is an open interval of \mathbb{R} containing 0, such that $c(0) = p$. We say that two curves $c_1, c_2 \in \mathcal{C}_p$ are equivalent (and we write $c_1 \sim c_2$) if there exists a coordinate neighborhood (U, Ω, ϕ) of p such that $(\phi \circ c_1)'(0) = (\phi \circ c_2)'(0)$ ¹, where $'$ denotes the usual derivative of a function from an open interval of \mathbb{R} containing 0 to an open subset of a Euclidean space. Then we define the *tangent space* $T_p M$ as the quotient of \mathcal{C}_p with respect of the equivalence relation \sim .

It is possible to show that $T_p M$ is actually a d -dimensional \mathbb{R} -vector space and that its elements act as “derivatives” on real valued functions defined on neighborhoods of p . To be precise by “derivative” we mean the following: let f be a smooth real valued function defined at least on a neighborhood of $p \in M$ and let v be an element of the tangent space $T_p M$; we define the application of v to f as the real number $(f \circ c)'(0)$ where c is any of the curves in the equivalence class v . To see how this works refer to [Ish99, Chap. 2]: there the tangent space is seen both as a “set of derivatives” and as a set of equivalence classes of curves (as in the above definition) and the equivalence of this two approaches is thoroughly analyzed.

Remark 1.1.4. Thanks to the \mathbb{R} -vector structure of $T_p M$, it is possible to introduce the *cotangent space* $T_p^* M$ as its dual: We define the elements of $T_p^* M$ as linear maps from $T_p M$ to \mathbb{R} . We obtain again a d -dimensional \mathbb{R} -vector space and then we can build via tensor products a new d^{i+j} -dimensional \mathbb{R} -vector space called *tensor space* of type (i, j) :

$$T_p^{(i,j)} M = (T_p M)^{\otimes i} \otimes (T_p^* M)^{\otimes j}.$$

¹Here the composition \circ is to be intended in a proper sense: $\phi \circ c_1$ denotes the composition of ϕ with a function d_1 from an open interval J of \mathbb{R} containing 0 (eventually smaller than the domain I of c_1) to U defined by $d_1(t) = c_1(t)$ for each $t \in I$.

By convention we set $T_p^{(0,0)}M = \mathbb{R}$. Finally we build the *tensor space* via direct sum:

$$\mathcal{T}_p M = \bigoplus_{(i,j) \in \mathbb{N} \times \mathbb{N}} T_p^{(i,j)} M.$$

This is again a real vector space (this time $\dim \mathcal{T}_p M = \infty$) and it can be even shown that $(\mathcal{T}_p M, \otimes)$ is an associative algebra generated by \mathbb{R} , $T_p M$ and $T_p^* M$.

Once that we have the notion of tangent space, we can define the *tangent bundle* TM of a manifold M as the disjoint union on the manifold of the tangent spaces at each point:

$$TM = \bigsqcup_{p \in M} T_p M.$$

Similarly we define the *cotangent bundle* T^*M , the *tensor bundle of type (i, j)* $T^{(i,j)}M$ and the *tensor bundle* $\mathcal{T}M$. Notice that $T^{(0,0)}M$ is simply $M \times \mathbb{R}$.

At this point $T^{(i,j)}M$ are merely sets. Hereafter we will endow them with a far richer structure.

Our knowledge about tangent spaces allows us to introduce a notion of differential at a point that can be patched on the entire manifold giving rise to the so called pushforward. This new differential at a fixed point indeed reduces to the usual differential when the manifolds involved are open subsets of Euclidean spaces endowed with the trivial atlas (the canonical identification of each tangent space at a point of an open subset of a Euclidean space with the same Euclidean space is understood).

Definition 1.1.5. Let M and N be two manifolds. Consider a smooth map $f : M \rightarrow N$ and a point $p \in M$. We define the *differential of f at p* as the map

$$\begin{aligned} d_p f : T_p M &\rightarrow T_{f(p)} N, \\ [c] &\mapsto [f \circ c], \end{aligned}$$

where $[\cdot]$ denotes the equivalence class in the appropriate tangent space that has \cdot as representative.

We define the *push-forward through f* as the map $f_* : TM \rightarrow TN$ such that $f_*|_{T_p M} = (p, d_p f)$ for each $p \in M$.

Moreover we say that f is:

- an *immersion* if $\dim M \leq \dim N$ and $d_p f$ is injective for each $p \in M$;
- a *submersion* if $\dim M \geq \dim N$ and $d_p f$ is surjective for each $p \in M$;
- an *embedding* if it is an immersion and f maps M homeomorphically onto its

image $f(M)$ (endowed with the topology induced by that of N), i.e. the map

$$\begin{aligned} f' : M &\rightarrow f(M) \\ p &\mapsto f(p) \end{aligned}$$

is a homeomorphism.

It is possible to show that the definition of differential at a point is well posed and it is easy to see that it reduces to the usual notion of differential when M and N are open subsets of Euclidean spaces, as anticipated. For this reason often the push-forward through f is also called *differential* and is denoted with df . Instead the name “push-forward” is due to the fact that in some sense f_* “pushes” through f each element $v \in TM$ to an element $f_*v \in TN$ in such a way that if $v \in T_pM$ then $f_*v \in T_{f(p)}N$.

Embeddings allow us to recognize submanifolds.

Definition 1.1.6. Let M be a manifold and let S be a manifold whose underlying set is included in M . We say that a manifold S is a *submanifold of M* if the inclusion map $\iota_S^M : S \rightarrow M$, $p \mapsto p$ is an embedding from S to M .

Remark 1.1.7. An important example of submanifold of a given d -dimensional manifold M is the following. Suppose that S is a connected open subset of M . We can endow S with the topology induced by the topology of M and we immediately recognize that S is a connected Hausdorff topological space with a countable basis of open subsets. We can define a coordinate neighborhood for S taking a coordinate neighborhood (U, Ω, ϕ) for M : We take $U \cap S$ as open subset of S (notice that this is also an open subset of M) and we use the fact that ϕ is a homeomorphism from U to Ω to deduce that we can take $\phi(U \cap S) \subseteq \Omega$ as open subset of \mathbb{R}^d . Then we define $\phi' : U \cap S \rightarrow \phi(U \cap S)$, $p \mapsto \phi(p)$ and we observe that ϕ' is a homeomorphism (it is bijective by construction and it is continuous with its inverse as a consequence of the same property for ϕ). Hence $(U \cap S, \phi(U \cap S), \phi')$ is a coordinate neighborhood for S (if it happens that $\phi(U \cap S)$ is not a neighborhood of 0, a translation in \mathbb{R}^d is sufficient to satisfy also this requirement). Applying this construction to all the elements of the maximal atlas of M , we obtain the maximal atlas of S and we recognize that S is actually a d -dimensional manifold. The inclusion map ι_S^M is smooth because the coordinate neighborhoods for S are the restrictions (in the sense of the construction above) of the coordinate neighborhoods for M and the transition charts of M are smooth by definition of manifold. For each $p \in S$, $d_p \iota_S^M$ is injective because each curve contained in a neighborhood of p in S is mapped through ι_S^M to the same curve in the same neighborhood of p , regarded now as a neighborhood with respect to the topology of M . This shows that ι_S^M is an immersion. Consider now the map $\iota_S^{M'} : S \rightarrow \iota_S^M(S) = S$, $p \mapsto \iota_S^M(p) = p$. If on the image $\iota_S^{M'}(S)$ we consider the topology that is induced by the topology of M , we realize that the topological space

$\iota_S^{M'}(S)$ coincides with the topological space S , hence it is trivial to check that $\iota_S^{M'}$ is a homeomorphism because it is nothing but the identity map of S . Then we realize that the d -dimensional manifold S constructed above is also a submanifold of M . Moreover $\iota_S^{M'}$ is a diffeomorphism as a consequence of the fact that all the transition charts for S are diffeomorphisms (this being a consequence of the existence of a maximal atlas for S). Moreover notice that ι_S^M is an open map because S is an open subset of M : Take an open subset Ω of S and note that trivially $\iota_S^M(\Omega) = \Omega$; since the topology on S is induced by that of M , we find an open subset Ω' of M such that $\Omega = \Omega' \cap S$; we deduce that Ω is also an open subset of M and we conclude that ι_S^M maps open subsets of S to open subsets of M , i.e. it is an open map.

A special case of this situation is the following. Let M and N be d -dimensional manifolds and suppose that $f : M \rightarrow N$ is an embedding whose image $f(M)$ is an open subset of N . Notice that $f(M)$ is also connected: We can find a curve contained in $f(M)$ connecting two arbitrary points p and q of $f(M)$ composing f with a curve γ in M that connects the preimages of p and q (the existence of γ follows from the hypothesis of connectedness of M). Applying the construction given above to the connected open subset $f(M)$ of N , we realize that $f(M)$ becomes a d -dimensional manifold that is a submanifold of N . Since f is an embedding, we have that f' is a homeomorphism. Now also $f(M)$ is a manifold so that we can ask whether f' has some more regularity beyond the continuity of itself and its inverse. To this end consider a point $p \in M$. We take a coordinate neighborhood (U, Ω, ϕ) of p in M and a coordinate neighborhood (V, Ω, ψ) of $f'(p)$ in $f(M)$. We recognize immediately that (V, Ω, ψ) is also a coordinate neighborhood of $f(p)$ in N because V , being an open neighborhood of $f'(p) = f(p)$ in the topology of $f(M)$, is also an open neighborhood of $f(p)$ in the topology of N . Recalling Definition 1.1.2, we have that $f_{U,V}$ is smooth by hypothesis and that $f'_{U,V} = f_{U,V}$ because f' and f coincide on $U \cap f^{-1}(V)$. Hence $f'_{U,V}$ is smooth too and the arbitrariness in the choice of the point $p \in M$ and of the coordinate neighborhoods implies that f' is smooth. On the one hand $d_p f'$ is injective for each $p \in M$ because f' is injective. On the other hand $d_p f'$ must be also surjective otherwise $\dim f(M) > \dim M$. Then the inverse function theorem implies that f' is a diffeomorphism. Using the inclusion map $\iota_{\psi(M)}^N : \psi(M) \rightarrow N$ (that is actually an embedding, as we saw above), we can decompose ψ in $\psi = \iota_{\psi(M)}^N \circ \psi'$. In particular this implies that ψ is an open map because ψ' is a homeomorphism and $\iota_{\psi(M)}^N$ is an open map as seen above.

Exploiting the definition of the cotangent space as dual of the tangent space, we can introduce a “dual” of the notion of push-forward.

Definition 1.1.8. Let M and N be two manifolds and let $f : M \rightarrow N$ be a smooth function. We call *pull-back through f* the map $f^* : \bigsqcup_{q \in f(M)} T_q^* N \rightarrow T^* M$ defined as the pointwise dual of the push-forward $f_* : TM \rightarrow TN$, i.e. for each $p \in M$, each

$\omega \in T_{f(p)}^*N$ and each $v \in T_pM$ we require that

$$\left(f^*|_{T_{f(p)}^*N} \omega\right) v_p = \omega \left(f_*|_{T_pM} v_{f(p)}\right),$$

where the dual pairings between the vector spaces T_p^*M and T_pM and between the vector spaces $T_{f(p)}^*N$ and $T_{f(p)}N$ are taken into account.

The reader should bear in mind that the dual pairing between T_p^*M and T_pM is actually part of the definition of T_p^*M as the vector space dual to T_pM (recall the definition of cotangent space in Remark 1.1.4).

Remark 1.1.9. Let M and N be two manifolds and let $f : M \rightarrow N$ be a smooth function. An extension of the notions of push-forward and pull-back is possible using the tensor structure of $T_p^{(i,j)}M$:

- the push-forward $f_* : T^{(i,0)}M \rightarrow T^{(i,0)}N$ through f is defined by

$$f_*|_{T_p^{(i,0)}M} (v_1 \otimes \cdots \otimes v_i) = f_*|_{T_pM} v_1 \otimes \cdots \otimes f_*|_{T_pM} v_i,$$

for each $p \in M$ and each $v_1, \dots, v_i \in T_pM$, where f_* on the RHS² is the push forward through f from TM to TN ;

- the pull-back $f^* : \bigsqcup_{q \in f(M)} T_q^{(0,j)}N \rightarrow T^{(0,j)}M$ through f , defined by

$$f^*|_{T_{f(p)}^{(0,j)}N} (\omega_1 \otimes \cdots \otimes \omega_j) = f^*|_{T_{f(p)}^*N} \omega_1 \otimes \cdots \otimes f^*|_{T_{f(p)}^*N} \omega_j,$$

for each $p \in M$ and each $\omega_1, \dots, \omega_j \in T_{f(p)}^*N$, where f^* on the RHS is the pull-back through f from $\bigsqcup_{q \in f(M)} T_q^*N$ to T^*M .

We can enlarge the notion of push-forward and pull-back much more if we suppose that $f : M \rightarrow N$ is a diffeomorphism: in such case the smooth map f is bijective and we have at our disposal also the smooth bijective map $f^{-1} : N \rightarrow M$, hence we can push forward through f^{-1} all the elements of $T^{(i,0)}N$ to $T^{(i,0)}M$ and we can pull back through f^{-1} all the elements of $T^{(0,j)}M$ to $T^{(0,j)}N$. This allows us to define a new push-forward and a new pull-back through f :

- the push-forward $f_* : T^{(i,j)}M \rightarrow T^{(i,j)}N$ through f is defined by

$$f_*|_{T_p^{(i,j)}M} (v \otimes \omega) = f_*|_{T_p^{(i,0)}M} v \otimes (f^{-1})^*|_{T_p^{(0,j)}M} \omega,$$

for each $p \in M$, each $v \in T_p^{(i,0)}M$ and each $\omega \in T_p^{(0,j)}M$, where on the RHS f_* denotes the push-forward through f from $T^{(i,0)}M$ to $T^{(i,0)}N$, while $(f^{-1})^*$ denotes the pull-back through f^{-1} from $T^{(0,j)}M$ to $T^{(0,j)}N$;

²Here, and in the rest of this thesis, the acronym “LHS” stands for “left hand side”, while the acronym “RHS” stands for “right hand side”.

- the pull-back $f^* : T^{(i,j)}N \rightarrow T^{(i,j)}M$ through f is defined by

$$f^*|_{T_q^{(i,j)}N}(v \otimes \omega) = (f^{-1})_*|_{T_q^{(i,0)}N}v \otimes f^*|_{T_q^{(0,j)}N}\omega,$$

for each $q \in N$, each $v \in T_q^{(i,0)}N$ and each $\omega \in T_q^{(0,j)}N$, where on the RHS $(f^{-1})_*$ denotes the push-forward through f^{-1} from $T^{(i,0)}N$ to $T^{(i,0)}M$, while f^* denotes the pull-back through f from $T^{(0,j)}N$ to $T^{(0,j)}M$.

In this way both f_* and f^* are extended to the whole tensor bundles over the appropriate manifolds. It turns out that this new $f_* : \mathcal{T}M \rightarrow \mathcal{T}N$ and $f^* : \mathcal{T}N \rightarrow \mathcal{T}M$ are inverses of each other.

Remark 1.1.10. Suppose that M is a d -dimensional manifold. Then our knowledge about push-forwards and pull-backs through smooth functions between manifolds allows us to recognize a manifold structure in $T^{(i,j)}M$. We give a sketch of how this is done for the case of TM (all other cases are similar). First of all we need a topology on tangent spaces. The fact that T_pM is a d -dimensional \mathbb{R} -vector space allows us to naturally identify it with \mathbb{R}^d . Using this identification we can also induce on each T_pM the usual topology of \mathbb{R}^d . TM becomes a topological space when endowed with the topology naturally induced by the disjoint union. Then we notice that this topology is Hausdorff and it admits a countable basis of open subsets as a consequence of the topologies on M and on each of the tangent spaces T_pM . Moreover TM is connected because M and all its tangent spaces are connected. Now we choose $v \in TM$. Since TM is the disjoint union over M of the tangent spaces T_pM , v is of the form (p, u) for some $p \in M$ and some $u \in T_pM$. We consider a coordinate neighborhood (U, Ω, ϕ) and we keep in mind that ϕ is a diffeomorphism (this follows from the maximality of the atlas of M). Then we take $V = \bigsqcup_{q \in U} T_qM$ and we realize that this is indeed a neighborhood of v in the topology of TM . Furthermore $T_qU = T_qM$ for each $q \in U$ since U is an open neighborhood of each $q \in U$ with respect to the topology of M , hence $TU = V$. With the identification of each $T_{\phi(q)}\Omega$ with \mathbb{R}^d , we have that

$$\phi_*(q, w) = (\phi(q), (d_q\phi)w) \in \{\phi(q)\} \times \mathbb{R}^d$$

for each $(q, w) \in V$ by definition of ϕ_* . We take $\Theta = \bigsqcup_{q \in U} T_{\phi(q)}\Omega$ and the above identification implies $\Theta = \Omega \times \mathbb{R}^d$ (notice that on $\Omega \times \mathbb{R}^d$ we consider the topology induced by the disjoint union otherwise the identification is not a homeomorphism). Considering ϕ_* as a map from $TU = V$ to $\Theta = \Omega \times \mathbb{R}^d$, we easily conclude that ϕ_* is a homeomorphism. Therefore (V, Θ, ϕ_*) is a coordinate neighborhood of v in TM . All transition maps are immediately diffeomorphisms (in the sense of functions between Euclidean spaces) and the maximal atlas of TM is easily built starting from the maximal atlas of M .

From this observation we can deduce that $f_* : T^{(i,j)}M \rightarrow T^{(i,j)}N$ and $f^* :$

$T^{(i,j)}N \rightarrow T^{(i,j)}M$ are diffeomorphisms between the manifolds $T^{(i,j)}M$ and $T^{(i,j)}N$ if $f : M \rightarrow N$ is a diffeomorphism. We show this fact in the case of TM , but the same proof works for any other tensor bundle of type (i, j) . As a matter of fact it suffices to show that both $f_* : TM \rightarrow TN$ and $f^* : TN \rightarrow TM$ are smooth functions between the manifolds TM and TN since, as we had already observed in Remark 1.1.9, f_* and f^* are inverses of each other. We focus on f_* . Suppose that O is an open subset of TN . Then O is of the form

$$O = \bigsqcup_{q \in \Omega} \Omega_q = \{(q, w) : q \in \Omega, w \in \Omega_q\},$$

where Ω is an open subset of N and Ω_q is an open subset of T_qN for each $q \in N$. Then we have that

$$\begin{aligned} (f_*)^{-1}(\Omega) &= f^*(\{(q, w) : q \in \Omega, w \in \Omega_q\}) \\ &= \left\{ \left(f^{-1}(q), f^*|_{T_qN} w \right) : q \in \Omega, w \in \Omega_q \right\} \\ &= \left\{ (p, v) : p \in f^{-1}(\Omega), v \in f^*|_{T_{f(p)}N}(\Omega_{f(p)}) \right\} \\ &= \bigsqcup_{p \in f^{-1}(\Omega)} f^*|_{T_{f(p)}N}(\Omega_{f(p)}). \end{aligned}$$

$f^{-1}(\Omega)$ is an open subset of M because f is continuous. Since for each $p \in M$ the map $f^*|_{T_pM}$ is linear between the finite dimensional topological vector spaces T_pM and $T_{f(p)}N$, it must be continuous too. Then it follows that

$$f^*|_{T_{f(p)}N}(\Omega_{f(p)}) = \left(f^*|_{T_pM} \right)^{-1}(\Omega_{f(p)})$$

is an open subset of T_pM for each $p \in \Omega$. We conclude that $(f_*)^{-1}(\Omega)$ has exactly the shape of an open subset of TM and this implies that f_* is continuous. Similarly we see that f^* is continuous and hence both f_* and f^* are homeomorphisms. Finally the smoothness of these maps easily follows from the smoothness of f and f^{-1} .

1.1.2 Vector bundles, connections and inner products

Till this point we have spoken of $T^{(i,j)}M$ as a manifold. However it is possible to recognize a richer structure on it. This structure is a special case of that presented in the next definition.

Definition 1.1.11. A *vector bundle* of rank n over a manifold of dimension d is a triple (E, M, π) , where E , called *total space*, and M , called *base*, are manifolds of dimension respectively $n + d$ and d and $\pi : E \rightarrow M$, called *projection*, is a smooth surjective map such that the following conditions hold:

- for each $p \in M$ the set $E_p = \pi^{-1}(p)$, called *fiber*, carries the structure of an

n -dimensional \mathbb{R} -vector space;

- for each $p \in M$ there exists a pair (U, Φ) , called *local trivialization at p* of (E, M, π) , where U is an open neighborhood of p in M and $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ is a diffeomorphism such that
 - $\text{pr}_1(\Phi(\mu)) = \pi(\mu)$ for each $\mu \in \pi^{-1}(U)$, where pr_1 denotes the projection on the first factor of the Cartesian product,
 - for each $q \in U$ the map $\Phi_q : E_q \rightarrow \{q\} \times \mathbb{R}^n$, defined by $\Phi_q \mu = \Phi(\mu)$ for each $\mu \in E_q$, is linear and bijective.

Note that the projection π is an open map, i.e. it maps open sets to open sets. This property is a consequence of the fact that the projection on an argument of a Cartesian product is always an open map.

Usually we will denote vector bundles only with their total space. However the choice of a base space and a projection is always understood.

The vectorial structure of each fiber allows us to construct other vector bundles using the vectorial operations (for example duality, tensor product, direct sum) fiberwise, provided that the vector bundles involved share the same base manifold. For example we can define the dual vector bundle E^* of the vector bundle E simply taking the dual spaces (in the usual sense of vector spaces) of the original fibers. Notice that the direct sum $E \oplus F$ of the vector bundles E and F is called *Whitney sum*.

Remark 1.1.12. bear in mind that each tensor bundle of type (i, j) can be endowed with a vector bundle structure. For example, in the case of the tangent bundle TM this is done considering TM as total space, M as base, the projection $\pi : TM \rightarrow M$ naturally induced by the disjoint union of tangent spaces as the projection, and $\{(U_\alpha, \phi_{\alpha*})\}$ as local trivializations (identification of $T\Omega$ with $\Omega \times \mathbb{R}^d$ is understood), where $\{(U_\alpha, \Omega_\alpha, \phi_\alpha)\}$ is the maximal atlas of M and $d = \dim M$. Then from now on, when we speak of $T^{(i,j)}M$, we refer to it as endowed with their natural vector bundle structure.

Notice that each tensor bundle of type (j, i) is the dual of the tensor bundle of type (i, j) and also the Whitney sum of j copies of TM and i copies of T^*M .

We can define maps between vector bundles that respect the vector bundle structures.

Definition 1.1.13. Let (E, M, π) and (F, N, σ) be two vector bundles. We call *vector bundle homomorphism* the pair (ψ, Ψ) where ψ is a smooth function from the base manifold M to the base manifold N and Ψ is a smooth function from the total space E to the total space F such that the following conditions hold:

- *compatibility with projections:* $\psi \circ \pi = \sigma \circ \Psi$;

- *fiberwise linearity*: Ψ is fiberwise a vector space homomorphism, i.e. the map

$$\begin{aligned}\Psi_p : E_p &\rightarrow F_{\psi(p)} \\ \mu &\rightarrow \Psi(\mu)\end{aligned}$$

is linear for each $p \in M$.

Then we say that (ψ, Ψ) is a *vector bundle isomorphism* if it is a bijective vector bundle homomorphism whose inverse is still a vector bundle homomorphism such that Ψ .

In the next remark we show a construction that allows to build a vector bundle whose total space and base space are submanifolds of the total space and the base space of a given vector bundle. We didn't include such construction immediately after the definition of vector bundles because we wanted to show also that the inclusion maps of the base space and of the total space as submanifolds give rise to a vector bundle homomorphism.

Remark 1.1.14. Suppose that a vector bundle E of rank n over a d -dimensional manifold M is given and assume that S is a connected open subset of M . In Remark 1.1.7 we saw that it is possible to use the manifold structure of M to make S a d -dimensional manifold itself. We also recognized that the new manifold S is a submanifold of M and that the inclusion map ι_S^M is an embedding of S into M . Now we consider the subset $\pi^{-1}(S)$ of the $(n+d)$ -dimensional manifold E . Since S is an open subset of M and π is continuous, $\pi^{-1}(S)$ is an open subset of E . One can check by contradiction that $\pi^{-1}(S)$ is connected exploiting the following properties: π is an open map, E is locally trivial and S is connected. Since S is connected Then it is possible to apply Remark 1.1.7 to the connected open subset $\pi^{-1}(S)$ of the manifold E . In this way we obtain a new $(n+d)$ -dimensional manifold (which is actually a submanifold of E) that we denote with $E|_S$. We define the map $\pi|_S : E|_S \rightarrow S$, $\mu \mapsto \pi(\mu)$ and we note that its image is

$$\pi|_S(E|_S) = \pi(\pi^{-1}(S)) = S,$$

hence $\pi|_S$ is surjective. Since the topologies and the atlases of the manifolds S and $E|_S$ are inherited via restriction from the topologies and the atlases of M and respectively E , it follows that $\pi|_S$ is continuous and also smooth. Then $(E|_S, S, \pi|_S)$ is our candidate to become a new vector bundle of rank n . The first thing to be checked is that $\pi|_S^{-1}(p)$ is an n -dimensional vector space for each $p \in S$: this fact is trivial because $\pi|_S^{-1}(p) = \pi^{-1}(p) = E_p$ and E_p is of course an n -dimensional vector space. It remains only the problem of the existence of local trivializations in neighborhoods of arbitrary points of S , but this difficulty is easily overcome in the following manner. Consider a point $p \in S$ and take a local trivialization (U, Φ) of

E at p . We note that $U \cap S$ is an open neighborhood of p in the topology of S and that we can define the map

$$\begin{aligned} \Phi' : \pi|_S^{-1}(U \cap S) = \pi^{-1}(U \cap S) &\rightarrow (U \cap S) \times \mathbb{R}^n \\ \mu &\mapsto \Phi(\mu) \end{aligned}$$

which satisfies

$$\text{pr}_1(\Phi'(\mu)) = \text{pr}_1(\Phi(\mu)) = \pi(\mu) = \pi|_S(\mu)$$

for each $\mu \in \pi|_S^{-1}(U \cap S)$ and is such that the map

$$\begin{aligned} \Phi'_p : \pi|_S^{-1}(p) = E_p &\rightarrow \{p\} \times \mathbb{R}^n \\ \mu &\mapsto \Phi'(\mu) = \Phi(\mu) = \Phi_p \mu \end{aligned}$$

is linear for each $p \in U \cap S$. These properties follow from the properties of Φ . Hence we have proved that for each point of S there exists a local trivialization. This implies that $(E|_S, S, \pi|_S)$ is a vector bundle in its own right. We will usually denote it simply with its total space $E|_S$ as it is customary for vector bundles.

Side by side with this construction, we can also introduce the inclusion maps $\iota_{E|_S}^E : E|_S \rightarrow E$, $\mu \mapsto \mu$ and $\iota_S^M : S \rightarrow M$, $p \mapsto p$. At this point we think E and $E|_S$ as $(n + d)$ -dimensional manifolds and we keep in mind that $E|_S$ is a submanifold of E . By definition of submanifold $\iota_{E|_S}^E : E|_S \rightarrow E$ is an embedding, hence, in particular, a smooth map. The same is true for ι_S^M . We note that $\pi_E \circ \iota_{E|_S}^E = \iota_S^M \circ \pi_{E|_S}$ and for each $p \in S$ we realize that the map

$$\begin{aligned} \iota_{E|_S p}^E : E|_{Sp} = E_p &\rightarrow E_{\iota_S^M(p)} = E_p \\ \mu &\mapsto \iota_{E|_S}^E(\mu) = \mu \end{aligned}$$

is a vector space isomorphism. These facts imply that $(\iota_S^M, \iota_{E|_S}^E)$ is a vector bundle homomorphism which is fiberwise a vector space isomorphism.

The following remark focuses the attention on vector bundle homomorphisms. It provides a procedure to restrict certain vector bundle homomorphisms to vector bundle isomorphisms.

Remark 1.1.15. Let E and F be vector bundles of rank n over d -dimensional manifolds M and N and consider a vector bundle homomorphism (ψ, Ψ) from E to F . Suppose that ψ is an embedding of M into N whose image $\psi(M)$ is open in N and that Ψ is fiberwise a vector space isomorphism. The first step is the application of the last part of Remark 1.1.7 from which we deduce that $\psi(M)$ is a d -dimensional submanifold of N and that the map $\psi' : M \rightarrow \psi(M)$, $p \mapsto \psi(p)$ is a diffeomorphism such that $\psi = \iota_{\psi(M)}^N \circ \psi'$. In particular we note that $\psi(M)$ is a connected open subset of N , hence it is possible to apply Remark 1.1.14 obtaining the new vector

bundle $F|_{\psi(M)}$. Defining the map $\Psi' : E \rightarrow F|_{\psi(M)}$, $\mu \mapsto \Psi(\mu)$, we can check that it is continuous with respect to the topologies of E and $F|_{\psi(M)}$ because Ψ is continuous with respect to the topologies of E and F and the topology on $F|_{\psi(M)}$, whose underlying set is an open subset of F , is induced by that of F . Moreover Ψ' is smooth because Ψ is smooth and the atlas of $F|_{\psi(M)}$ is nothing but the restriction of the atlas of F . We can even draw more accurate conclusions noting that

$$\pi_{F|_{\psi(M)}}(\Psi'(\mu)) = \pi_F(\Psi(\mu)) = \psi(\pi_E(\mu))$$

for each $\mu \in F|_{\psi(M)}$ and that for each $p \in M$ the map

$$\begin{aligned} \Psi'_p : E_p &\rightarrow F|_{\psi(M)\psi(p)} = F_{\psi(p)} \\ \mu &\mapsto \Psi'(\mu) = \Psi(\mu) = \Psi_p \mu \end{aligned}$$

is linear. This shows that (ψ', Ψ') is a vector bundle homomorphism. We assumed that $\Psi_p = \Psi'_p$ is a vector space isomorphism for each $p \in M$, hence Ψ' is exactly defined as the restriction of Ψ to its image (for this reason from now on we will denote the vector bundle $F|_{\psi(M)}$ with $\Psi(E)$). This shows that Ψ' is a bijective smooth function and that (ψ', Ψ') is a bijective vector bundle homomorphism. Some work with local trivializations and coordinate neighborhoods shows that for each $p \in M$ there exists an open neighborhood U of p in M such that the Jacobian determinant of Ψ' (locally trivialized and written in local coordinate) at p is not null. Hence also Ψ'^{-1} is a smooth function between the manifolds $\Psi(E)$ and E as a consequence of the inverse function theorem. It is easy to check that

$$\pi_E \circ \Psi'^{-1} = \psi'^{-1} \circ \pi_{\Psi(E)}$$

and that for each $q \in \psi(M)$ the map

$$\begin{aligned} \Psi'^{-1}_q : \Psi(E)_q &\rightarrow E_{\psi'^{-1}(q)} \\ \nu &\mapsto \Psi'^{-1}(\nu) \end{aligned}$$

coincides with the inverse of $\Psi'_{\psi'^{-1}(q)}$ (hence, in particular, it is a vector space homomorphism). Then we conclude that (ψ'^{-1}, Ψ'^{-1}) is a vector bundle homomorphism from the vector bundle $\Psi(E)$ to the vector bundle E and that it is the inverse of (ψ', Ψ') so that (ψ', Ψ') is a vector bundle isomorphism. Remark 1.1.14 provides a vector bundle homomorphism $(\iota_{\psi(M)}^N, \iota_{\Psi(E)}^F)$ from $\Psi(E)$ to F which is fiberwise a vector space isomorphism. This gives us the opportunity to decompose the original vector bundle homomorphism (ψ, Ψ) : We already know that $\psi = \iota_{\psi(M)}^N \circ \psi'$ and it can be directly checked that $\Psi = \iota_{\Psi(E)}^F \circ \Psi'$, hence we deduce that

$$(\psi, \Psi) = (\iota_{\psi(M)}^N \circ \psi', \iota_{\Psi(E)}^F \circ \Psi') = (\iota_{\psi(M)}^N, \iota_{\Psi(E)}^F) \circ (\psi', \Psi').$$

Now we want to introduce a particular class of smooth functions from a manifold to the total space of a vector bundle whose base is such manifold. The peculiarity of such maps resides in their compatibility with the projection of the vector bundle.

Definition 1.1.16. Let E be a vector bundle over a manifold M . A C^k -section in E is a C^k -function s from the base manifold M to the total space manifold E such that $\pi \circ s = \text{id}_M$.

A (smooth) section in E is a C^k -section in E for each k or, equivalently, is a smooth function s from the base manifold M to the total space manifold E such that $\pi \circ s = \text{id}_M$.

The space of C^k -sections $C^k(M, E)$ is the set comprised by all C^k -sections in E , the space of (smooth) sections $C^\infty(M, E)$ is the set comprised by all the smooth sections in E and finally the space of smooth sections with compact support $\mathcal{D}(M, E)$ (or $C_0^\infty(M, E)$) is the set comprised by all the smooth sections in E with compact support.

Note that if s is a smooth section on a vector bundle, we will often simply say that s is a section. On the contrary for C^k -sections we will never omit the prefix C^k . We observe that the fiberwise vector structure of each vector bundle induces a vector structure on the set of sections in such vector bundle. This fact motivates the word “space” (in the sense of vector space) used to denote the set of C^k -sections, the set of sections and the set of compactly sections defined above.

Remark 1.1.17. Let E and F be vector bundles over the manifolds M and respectively N and let s be a section in E . Consider a vector bundle homomorphism (ψ, Ψ) from E to F , where ψ is an embedding of the manifold M into the manifold N whose image $\psi(M)$ is an open subset of N and Ψ is fiberwise a vector space isomorphism. Then we can apply Remark 1.1.15 and use the vector bundle isomorphism (ψ', Ψ') from E to $\Psi(E)$ to define the function $\Psi' \circ s \circ \psi'^{-1}$ from $\psi(M)$ to $\Psi(E)$. This is undoubtedly a smooth map because it is a composition of smooth maps and we can ask whether it is a section in the vector bundle $\Psi(E)$. The answer is positive because

$$\pi_{\Psi(E)} \circ \Psi' \circ s \circ \psi'^{-1} = \psi' \circ \pi_E \circ s \circ \psi'^{-1} = \psi' \circ \text{id}_M \circ \psi'^{-1} = \text{id}_{\psi(M)}.$$

Note that, when (ψ, Ψ) is a vector bundle isomorphism from E to F , we can directly use it to obtain the section $\Psi \circ s \circ \psi^{-1}$ in F from a section s in E and its inverse (ψ^{-1}, Ψ^{-1}) to obtain the section $\Psi^{-1} \circ t \circ \psi$ from a section t in F . With an abuse of language we say that $\Psi \circ s \circ \psi^{-1}$ and $\Psi^{-1} \circ t \circ \psi$ are respectively the push-forward of s and the pull-back of t through (ψ, Ψ) .

From Remark 1.1.9 we can deduce that, given a diffeomorphism f between the manifolds M and N , (f, f_*) can be recognized as a vector bundle homomorphism between the tangent bundles TM and TN (intended as vector bundles) and similarly

(f^{-1}, f^*) can be recognized as a vector bundle homomorphism between the tangent bundles T^*N and T^*M . Moreover we can extend them to tensor bundles of arbitrary type respectively over M and N and realize that $(f, f_*) : T^{(i,j)}M \rightarrow T^{(i,j)}N$ and $(f^{-1}, f^*) : T^{(i,j)}N \rightarrow T^{(i,j)}M$ are inverses of each other so that are both vector bundle isomorphisms. These observations allow us to push forward and pull back sections in tensor bundles of any type through diffeomorphisms of the base manifolds exactly as we do with vector bundle isomorphisms.

Example 1.1.18. A simple example of a space of sections is provided by the set $C^\infty(M)$ of smooth real valued functions over the manifold M . As a matter of fact in such case we can identify each $f \in C^\infty(M)$ with the map

$$\begin{aligned} M &\rightarrow M \times \mathbb{R} \\ p &\mapsto (p, f(p)) \end{aligned}$$

(still called f) which is immediately recognized as a section in the trivial tensor bundle $T^{(0,0)}M = M \times \mathbb{R}$.

We take the chance to introduce some nomenclature: Sections in the tangent bundle of a manifold are usually called *vector fields*, while sections in the cotangent bundle are known as *1-forms*. Moreover sections in each tensor bundle of type (i, j) are generally called *tensor fields*.

In a vector bundle there is no natural notion of differentiation, so that we must provide such notion together with the vector bundle in order to be able to do calculus.

Definition 1.1.19. Let M be a manifold and let E be a vector bundle over M . A (linear) connection on E is a map

$$\begin{aligned} \nabla : C^\infty(M, TM) \times C^\infty(M, E) &\rightarrow C^\infty(M, E) \\ (X, s) &\mapsto \nabla_X s \end{aligned}$$

that satisfies the following properties:

- $C^\infty(M, \mathbb{R})$ -linearity in the first argument: for each $f, h \in C^\infty(M)$, each $X, Y \in C^\infty(M, TM)$ and each $s \in C^\infty(M, E)$ it holds

$$\nabla_{(fX+gY)}s = f\nabla_X s + g\nabla_Y s;$$

- \mathbb{R} -linearity in the second argument: for each $a, b \in \mathbb{R}$, each $X \in C^\infty(M, TM)$ and each $s, t \in C^\infty(M, E)$ it holds

$$\nabla_X(as + bt) = a\nabla_X s + b\nabla_X t;$$

- *Leibniz rule in the second argument:* for each $s \in C^\infty(M, E)$, each $f \in C^\infty(M)$ and each $X \in C^\infty(M, TM)$ it holds that

$$\nabla_X(fs) = (\partial_X f)s + f\nabla_X s,$$

where $\partial_X f$ is the section in TM defined by $(\partial_X f)(p) = (d_p f)(X(p))$ for each $p \in M$.

The properties required allow us to think a connection ∇ as a map

$$C^\infty(M, TM) \otimes C^\infty(M, E) = C^\infty(M, TM \otimes E) \rightarrow C^\infty(M, E)$$

or also as a map

$$C^\infty(M, E) \rightarrow C^\infty(M, T^*M) \otimes C^\infty(M, E) = C^\infty(M, T^*M \otimes E).$$

We want to stress that on a given vector bundle there may be several possible inequivalent connections. This is indeed the case also for tensor bundles of each type. A concrete example of a connection on the trivial tensor bundle $T^{(0,0)}M = M \times \mathbb{R}$ is provided by the map

$$\partial : C^\infty(M, TM) \times C^\infty(M, M \times \mathbb{R}) \rightarrow C^\infty(M, M \times \mathbb{R})$$

defined in the statement of the Leibniz rule for a connection (the identification of $C^\infty(M)$ with the space of sections $C^\infty(M, M \times \mathbb{R})$ presented in Example 1.1.18 is understood).

Notice that there is a natural way to induce a connection on a vector bundle built through fiberwise vectorial operations (e.g. duality, tensor product and Whitney sum) starting from the connections on the original vector bundles. Examples are provided by the following formulas (we put superscripts on ∇ to indicate the vector bundle on which the connection is defined):

$$\begin{aligned} (\nabla_X^{E^*} \nu)(\mu) &= \partial_X(\nu(\mu)) - \nu(\nabla_X^E \mu), \\ \nabla_X^{E \otimes F}(\mu \otimes \rho) &= (\nabla_X^E \mu) \otimes \rho + \mu \otimes (\nabla_X^F \rho), \\ \nabla_X^{E \oplus F}(\mu \oplus \rho) &= (\nabla_X^E \mu) \oplus (\nabla_X^F \rho), \end{aligned}$$

where E and F are vector bundles over a manifold M endowed with connections ∇^E and respectively ∇^F , X is an arbitrary vector field over M and μ, ν and ρ are arbitrary sections respectively in E, E^* and F .

Now we want to define an object that characterizes the behavior of each connection on a given vector bundle. To do this we need the following construction. Suppose that ∇ is a connection over the vector bundle (E, M, π) and fix a point $p \in M$. We denote with d the dimension of M and with n the rank of E . There

exists a neighborhood U of p such that (U, Ω, ϕ) is a coordinate neighborhood of p in M and (U, Φ) is a local trivialization at p of E . On the one hand, using the coordinate neighborhood, we can obtain a set of local vector fields (i.e. sections in TE) $\{\partial_1, \dots, \partial_d\}$ that are pointwise linearly independent: This is done pushing forward through the diffeomorphism $\phi^{-1} : \Omega \rightarrow U$ the vector fields $\{v_1, \dots, v_d\}$ on $T\Omega$ (identified with $\Omega \times \mathbb{R}^d$) that are defined by $v_i(x) = e_i$ for each $x \in \Omega$ and each $i \in \{1, \dots, d\}$, where $\{e_1, \dots, e_d\}$ is the standard orthonormal base of \mathbb{R}^d . On the other hand, once chosen an orthonormal base $\{f_1, \dots, f_n\}$ of \mathbb{R}^n , we obtain a set of sections $\{\mu_1, \dots, \mu_n\}$ in $E|_U = \pi^{-1}(U)$ that are pointwise linearly independent setting $\mu_j(q) = \Phi^{-1}(q, f_j)$ for each $q \in U$ and each $j \in \{1, \dots, n\}$. That done, we can define the Christoffel symbols.

Definition 1.1.20. Let ∇ be a connection over the vector bundle (E, M, π) . With the construction given above, we can define the *Christoffel symbols* Γ_{ij}^k of the connection ∇ in a neighborhood U of a point p in M imposing $\Gamma_{ij}^k \mu_k = \nabla_{\partial_i} \mu_j$ (summation over k is implied).

Consider a vector bundle E over a manifold M endowed with a connection ∇ and fix a smooth curve $c : [a, b] \rightarrow M$ and $s_0 \in E_{c(a)}$. We can consider the following problem: Determine s from $[a, b]$ to E satisfying

$$\begin{aligned} \nabla_{X_t} s(t) &= 0 \quad \text{for } t \in [a, b], \\ s(a) &= s_0, \end{aligned}$$

where $X_t \in T_{c(t)}M$ is the vector tangent to c in $c(t)$. Written in local coordinates such problem reduces to a system of linear first order ordinary differential equations, hence the solution exists and is unique once that $s_0 \in E_{c(a)}$ is given. In particular we obtain $s(b)$. This allows us to give the next definition.

Definition 1.1.21. Let (E, M, π) be a vector bundle and let ∇ be a connection on it. For each smooth curve $c : [a, b] \rightarrow M$ we define the *parallel transport along c* as the linear function $Y_c : E_{c(a)} \rightarrow E_{c(b)}$ that maps each $s_0 \in E_{c(a)}$ to $s(b)$ as above.

We underline that in general the parallel transport depends upon the choice of the curve connecting its endpoints, but, once that a curve is chosen, the connection gives us a way to “connect” different fibers of the vector bundle through parallel transport.

We want to present another object that characterizes a connection on a vector bundle. However its definition requires a new tool.

Definition 1.1.22. Let M be a manifold. We call *Lie bracket* the map

$$[\cdot, \cdot] : C^\infty(M, TM) \times C^\infty(M, TM) \rightarrow C^\infty(M, TM)$$

uniquely determined by the following condition:

$$\partial_{[X,Y]}f = \partial_X\partial_Yf - \partial_Y\partial_Xf \quad \forall X, Y \in C^\infty(M, TM), \forall f \in C^\infty(M, M \times \mathbb{R}).$$

We take the chance to state the properties of the Lie bracket: it is \mathbb{R} -bilinear, antisymmetric and satisfies the *Jacobi identity*, i.e. for each $X, Y, Z \in C^\infty(M, TM)$ it holds that

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Now we are in position to properly define the curvature of a connection on a vector bundle.

Definition 1.1.23. Let (E, M, π) be a vector bundle endowed with a connection ∇ . We call *curvature of the connection* ∇ the map

$$C : C^\infty(M, TM) \times C^\infty(M, TM) \times C^\infty(M, E) \rightarrow C^\infty(M, E)$$

defined by

$$C(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]}s,$$

where $X, Y \in C^\infty(M, TM)$ and $s \in C^\infty(M, E)$.

Remark 1.1.24. From its definition, we deduce that C is \mathbb{R} -bilinear and antisymmetric in the first two arguments and \mathbb{R} -linear in the last argument. Therefore, denoting with \otimes_a the antisymmetrized tensor product, we can interpret C as a map from $C^\infty(M, TM) \otimes_a C^\infty(M, TM) \otimes C^\infty(M, E)$ to $C^\infty(M, E)$. Moreover its value at each point $p \in M$ depends only on the values of X, Y and s in an arbitrary neighborhood of p so that we are allowed to think C as a section in the vector bundle $T^*M \otimes_a T^*M \otimes E^* \otimes E$.

C can be locally written in components following a procedure analogous to that used to define Christoffel symbols (see before Definition 1.1.20): For a fixed point $p \in M$, we can find an open neighborhood U of p in M , a set of pointwise linearly independent local vector fields $\{v_1, \dots, v_d\}$ over U and a set of pointwise linearly independent local sections $\{\mu_1, \dots, \mu_n\}$ in $E|_U$, where we set $d = \dim M$ and $h = \text{rank} E$, and we can define C_{ijk}^l imposing $C_{ijk}^l \mu_l = C(v_i, v_j) \mu_k$. Using this definition it is possible to obtain the expression of C_{ijk}^l in terms of the Christoffel symbols and their derivatives ∂ along the local vector fields.

Now we define inner products on vector bundles. This additional structure allows us to pick out a specific connection on the tangent bundle of a manifold that has particular importance for General Relativity.

Definition 1.1.25. Consider a vector bundle E over the manifold M . We call *inner product on E* a section g in $E^* \otimes E^*$ that fulfils the following requirements:

- (fiberwise) symmetry: for each $p \in M$ and each $u, v \in E_p$ it holds that

$$g(p)(u \otimes v) = g(p)(v \otimes u);$$

- (fiberwise) non degeneracy: for each $p \in M$ we have the implication

$$u \in E_p : g(p)(u \otimes v) = 0 \forall v \in E_p \implies u = 0.$$

Inner products on TM are called *metrics on M* . *Riemannian metrics* are those whose signature is of type $(+, \dots, +)$ at any point, while *Lorentzian metrics* have signature of type $(-, +, \dots, +)$.

In some situations it is customary to define Lorentzian metrics with the requirement that their signature is of type $(+, -, \dots, -)$. We can pass from our definition to this one simply taking $-g$ in place of g .

Usually we will denote $g(p)(u \otimes v)$ with $u \cdot_{g,p} v$ and, if there is no risk of misunderstanding, we will also omit g in our notation so that $u \cdot_{g,p} v$ becomes $u \cdot_p v$. In the case of a metric we will write $g_p(u, v)$ in place of $g(p)(u \otimes v)$.

Remark 1.1.26. Notice that each inner product on E is automatically a (fiberwise) non degenerate, symmetric, bilinear form from $E \times E$ to the trivial vector bundle $M \times \mathbb{R}$. From another point of view, we could define inner products on a vector bundle E as (fiberwise) non degenerate sections in the vector bundle $E^* \otimes_s E^*$, where \otimes_s denotes the symmetrized tensor product. In this way the set of inner products on E becomes a subset of the vector space $C^\infty(M, E^* \otimes_s E^*)$, which becomes a Fréchet space when endowed with the usual topology of C^∞ sections.

With the usual procedure we can locally rewrite in components an inner product g on a vector bundle E . We must only fix $p \in M$, consider a local trivialization of E in an open neighborhood U of p , find a set $\{\mu_1, \dots, \mu_n\}$ of pointwise linearly independent sections in $E|_U$ and set $g_{ij}(q) = \mu_i \cdot_q \mu_j$ for each $q \in U$, where $n = \text{rank} E$. The property of fiberwise symmetry implies that $g_{ij}(q) = g_{ji}(q)$, while non degeneracy implies that $(g_{ij}(q))$ is an invertible $n \times n$ matrix. This holds for each $q \in U$. We denote by $(g^{ij}(q))$ the inverse of $(g_{ij}(q))$ for each $q \in U$.

Using an inner product on a vector bundle E we can define the so called musical isomorphisms between E and its dual E^* .

Definition 1.1.27. Let E be a vector bundle endowed with an inner product g . We define

- the *lowering isomorphism*:

$$\begin{aligned} \flat : E &\rightarrow E^*, \\ \mu &\mapsto g|_{\pi_E(\mu)}(\mu \otimes \cdot); \end{aligned}$$

- the *raising isomorphism*:

$$\sharp = \flat^{-1} : E^* \rightarrow E.$$

The raising isomorphism and the lowering isomorphism are collectively called *musical isomorphisms*.

As suggested by their names, \flat and \sharp are both vector bundle isomorphisms.

We anticipated that we can uniquely determine a specific connection on the tangent bundle of a manifold endowed with a metric.

Theorem 1.1.28. *Consider a manifold M endowed with a metric g . Then there exists a unique connection ∇ on TM that satisfies the following requirements:*

- ∇ is metric, i.e. for each $X, Y, Z \in TM$ it holds that

$$\partial_X (g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z);$$

- ∇ is torsion free, i.e. for each $X, Y \in TM$ it holds that

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

This connection ∇ on TM is called Levi-Civita connection.

From the requirements singling out the Levi-Civita connection among all possible connections on TM , we can determine the Christoffel symbols of the Levi-Civita connection (this fact actually guarantees uniqueness of the Levi-Civita connection). This is done by fixing a point $p \in M$ and choosing a coordinate neighborhood of p and a set of pointwise orthonormal (with respect to the metric on M) local vector fields $\{v_1, \dots, v_d\}$ ($d = \dim M$): we easily find

$$\Gamma_{ij}^k = g^{kl} \frac{1}{2} (\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}), \quad (1.1.1)$$

where ∂_i denotes ∂_{v_i} . Note that the symmetry $g_{ij} = g_{ji}$ implies that the Christoffel symbols of the Levi-Civita connection satisfy

$$\Gamma_{ij}^k = \Gamma_{ji}^k. \quad (1.1.2)$$

We stress that each time that we will encounter a connection on the tangent bundle of a manifold endowed with a metric, such connection will be the Levi-Civita one.

Till now we have dealt with the curvature of a connection on an arbitrary vector bundle. In the special case of the tangent bundle TM of a manifold M endowed with a metric g we can define other associated objects, namely the *Ricci tensor* R_{ij} and

the scalar curvature S . We define them locally starting from the local definitions of $C_{ijk}{}^l$ and g^{ij} : $R_{ik} = C_{ijk}{}^j$ and $S = g^{ij} R_{ij}$.

We present here the expressions of the curvature and of the Ricci tensor for the Levi-Civita connection on a manifold endowed with a metric:

$$\begin{aligned} C_{ijk}{}^l &= \partial_j \Gamma_{ik}^l - \partial_i \Gamma_{jk}^l + \Gamma_{ik}^m \Gamma_{jm}^l - \Gamma_{jk}^m \Gamma_{im}^l; \\ R_{ij} &= \partial_k \Gamma_{ij}^k - \partial_i \Gamma_{kj}^k + \Gamma_{ij}^l \Gamma_{kl}^k - \Gamma_{kj}^l \Gamma_{il}^k. \end{aligned} \quad (1.1.3)$$

1.1.3 Differential forms on a manifold

In this subsection we discuss a specific class of tensor fields over an arbitrary manifold M , called differential forms. A much more detailed discussion in this topic can be found in [Boo86, Chap. V].

Let M be a d -dimensional manifold and fix $p \in M$. For $k \in \mathbb{N}$, we consider $T_p^{(0,k)} M$. We would like to pick out a subspace $T_p^{(0,k)} M$, specifically the one consisting of such elements that are skew-symmetric when intended as k -linear maps from $T_p M \times \cdots \times T_p M$ (k times) to \mathbb{R} . To recognize these elements we need a new tool, the alternating map.

Definition 1.1.29. Let M be a manifold and consider $p \in M$ and $k \in \mathbb{N}$. We define the *alternating map at p*

$$\mathbf{a} : T_p^{(0,k)} M \rightarrow T_p^{(0,k)} M$$

setting for each $\omega \in T_p^{(0,k)} M$, each $v_1, \dots, v_k \in T_p M$

$$(\mathbf{a}\omega)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma} (\text{sgn} \sigma) \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}),$$

where σ is a permutation of $1, \dots, k$ and $\text{sgn} \sigma$ is its sign.

Using the alternating map \mathbf{a} defined just above, we can introduce alternating tensor bundles of type k over a manifold M .

Definition 1.1.30. Let M be a d -dimensional manifold and consider $p \in M$ and $k \in \mathbb{N}$. An *alternating tensor of type k over a manifold M at p* is an element $\omega \in T_p^{(0,k)} M$ such that $\mathbf{a}\omega = \omega$. The *alternating tensor space of type k* , denoted by $\Lambda_p^k M$, is the set of all alternating tensors of type k over a manifold M at p and the *alternating tensor bundle of type k* , denoted by $\Lambda^k M$, is the disjoint union over $p \in M$ of $\Lambda_p^k M$.

By convention we set $\Lambda_p^0 M = \mathbb{R}$ for each $p \in M$ and $\Lambda^0 M = M \times \mathbb{R}$.

It turns out that $\Lambda_p^k M$ is a real vector space for each $p \in M$ and each $k \in \mathbb{N}$ and that $\Lambda^k M$ is a vector bundle for each $k \in \mathbb{N}$. It can be shown that $\Lambda_p^k M = \{0\}$ for each $k > d$ and that $\dim(\Lambda_p^k M) = \binom{d}{k}$.

Once that a point $p \in M$ is fixed, it is possible to define a new algebra in a way similar to that followed for the definition of the tensor algebra $(\mathcal{T}_p M, \otimes)$. The underlying set of such algebra is

$$\Lambda_p M = \bigoplus_{k=0}^d \Lambda_p^k M.$$

The vector structure on $\Lambda_p M$ is naturally induced by the direct sum \oplus , while the algebraic structure requires the introduction of a new operation, the so called wedge product.

Definition 1.1.31. Let M be a d -dimensional manifold and consider $k, k' \in \mathbb{N}$ and $p \in M$. The wedge product \wedge is the map from $\Lambda_p^k M \times \Lambda_p^{k'} M \rightarrow \Lambda_p^{k+k'} M$ defined by the formula

$$\eta \wedge \xi = \binom{k+k'}{k} \mathbf{a}(\eta \otimes \xi)$$

for each $\eta \in \Lambda_p^k M$ and each $\xi \in \Lambda_p^{k'} M$.

It can be shown that \wedge can be naturally extended to an operation on $\Lambda_p M$ (still called wedge product and denoted by \wedge) that is binary, internal, bilinear and associative. This allows us to conclude that $(\Lambda_p M, \wedge)$ is an associative algebra. The fact that $\Lambda_p^k M = T_p^* M \wedge \cdots \wedge T_p^* M$ (k times) implies that $(\Lambda_p M, \wedge)$ is generated by \mathbb{R} and $T_p^* M$.

In addition to such pointwise algebraic structure, it is possible to define a new vector bundle ΛM through the disjoint union of $\Lambda_p M$ over $p \in M$. ΛM is called *alternating tensor bundle*. As a by product of this construction we obtain an extension of the wedge product to an operation on the alternating tensor bundle. In particular we have that

$$\Lambda^k M = \bigwedge^k T^* M \quad \forall k \quad \text{and} \quad \Lambda M = \bigoplus_{k=0}^d \Lambda^k M.$$

The above preparation allows us to define k -forms.

Definition 1.1.32. We say that a k -form over M (also called *differential form of order k over M*) is a section in the alternating tensor bundle of order k $\Lambda^k M$. The *space of k -forms over M* is denoted by $\Omega^k M$. We define the *space of differential forms over M* ΩM as the direct sum of all the non trivial spaces of k -forms.

Notice that $\Omega^0 M = C^\infty(M)$ and that $\Omega^k M = \{0\}$ for each $k > \dim M$. We easily recognize that $\Omega^k M$ is a vector spaces for each k . This fact motivates the word “space” (intended in the sense of vector space) used in the last definition.

Previously we defined the wedge product pointwisely. It is possible to extend this operation from the alternating tensor spaces at each point to the space of differential

forms ΩM simply imposing $(\Xi \wedge \Theta)(p) = \Xi(p) \wedge \Theta(p)$ for each $p \in M$ and each $\Xi, \Theta \in \Omega M$. It turns out that $(\Omega M, \wedge)$ is an associative algebra, known as *exterior algebra of M* .

As a consequence of its definition, $\Omega^k M$ is a subspace of $C^\infty(M, T^{(0,k)}M)$ for each $k \in \mathbb{N}$. This fact guarantees that the observations made about push-forwards and pull-backs through a diffeomorphism of sections in tensor bundles of any type (see Remark 1.1.9) applies also in this case, hence we can push forward and pull back any k -form using a diffeomorphism.

We take the chance to state some useful properties of the wedge product.

Proposition 1.1.33. *Let M and N be manifolds. Then for each $k, k' \in \mathbb{N}$ the wedge product \wedge fulfils the following properties:*

- $\Xi \wedge \Theta = (-1)^{kk'} \Theta \wedge \Xi$ for each $\Xi \in \Omega^k M$ and each $\Theta \in \Omega^{k'} M$;
- $(f\Xi) \wedge \Theta = f(\Xi \wedge \Theta)$ for each $\Xi \in \Omega^k M$, each $\Theta \in \Omega^{k'} M$ and each $f \in C^\infty(M)$;
- if $f : M \rightarrow N$ is a smooth function, for each $\Xi \in \Omega^k N$ and each $\Theta \in \Omega^{k'} N$

$$f^*(\Xi \wedge \Theta) = (f^*\Xi) \wedge (f^*\Theta),$$

where the wedge on the LHS is defined on ΩN , while the wedge on the RHS is defined on ΩM .

As a consequence of the last theorem we can conclude that, for each diffeomorphism $f : M \rightarrow N$, the vector bundle isomorphism $(f^{-1}, f^*) : \Lambda N \rightarrow \Lambda M$ induces an algebraic isomorphism between the exterior algebras ΩN and ΩM . Notice that one can similarly consider the vector bundle isomorphism $(f, f_*) : \Lambda M \rightarrow \Lambda N$ and conclude that this induces an algebraic isomorphism between the exterior algebras ΩM and ΩN . Moreover these algebraic isomorphisms are inverses of each other.

Thanks to the following theorem it is possible to define a new operation on the exterior algebra of M that is a sort of special case of the push-forward of a real valued smooth function (also called differential, see Definition 1.1.5).

Proposition 1.1.34. *For each manifold M there exists a unique \mathbb{R} -linear map $d_M : \Omega M \rightarrow \Omega M$, called exterior derivative, fulfilling the following properties:*

- the exterior derivative coincides with the differential on $\Omega^0 M = C^\infty(M)$, i.e. $d_M f = df$ for each $f \in \Omega^0 M = C^\infty(M)$;
- for each $\Xi \in \Omega^k M$ and each $\Theta \in \Omega^{k'} M$ it holds that

$$d_M(\Xi \wedge \Theta) = d_M \Xi \wedge \Theta + (-1)^k \Xi \wedge d_M \Theta;$$

- $d_M^2 = d_M \circ d_M = 0$.

Moreover the exterior derivative satisfies another property: For each smooth map f from a manifold M to a manifold N we have

$$f^* \circ d_N = d_M \circ f^*.$$

Notice that the last property of the exterior derivative may also be read in this way if $f : M \rightarrow N$ is a diffeomorphism:

$$f_* \circ d_M = d_N \circ f_*.$$

In the following we will denote the exterior derivative simply with d , omitting the subscript referred to the manifold. Notice that there is no risk of confusion between exterior derivative and differential because they coincide in the only situation in which they may be confused, that is $\Omega^0 M = C^\infty(M)$.

Using the exterior derivative, we can introduce a classification of k -forms and then define the de Rham cohomology groups that will be used to introduce an hypothesis when we will discuss the electromagnetic field.

Definition 1.1.35. Let M be a d -dimensional manifold and consider $k \in \{1, \dots, d\}$. We say that $\Theta \in \Omega^k M$ is *closed* if $d\Theta = 0$ while we say that it is *exact* if there exists $\Xi \in \Omega^{k-1} M$ such that $d\Xi = \Theta$. We also denote with $Cl^k(M)$ the space of closed k -forms over M and with $Ex^k(M)$ the space of exact k -forms over M .

We call k -th de Rham cohomology group of M the quotient space $H^k(M) = Cl^k(M)/Ex^k(M)$.

Notice that $Cl^k(M)$ and $Ex^k(M)$ are actually vector spaces because d is linear on ΩM and $Cl^k(M)$ is the kernel of d when restricted to $\Omega^k M$, while $Ex^k(M)$ is the image of $\Omega^{k-1} M$ through d . Moreover since $d^2 = 0$, $Ex^k(M) \subseteq Cl^k(M)$. Hence $H^k(M)$ is a well defined vector space.

In a d -dimensional manifold M , d -forms are of particular importance because we can use them to define the orientability and the orientation of a manifold. These notions will become relevant in the next subsection.

Definition 1.1.36. Let M be a d -dimensional manifold. We say that M is *orientable* if there exists a d -form Θ over M which is nowhere null. If M is orientable and Θ is a choice of a nowhere null d -form over M , we say that Θ fixes an *orientation* on M and we call M an *oriented manifold*.

Let M and N be two orientable manifolds and let $f : M \rightarrow N$ be an embedding. Choose a nowhere null d -form Θ over M and a nowhere null d -form Ξ over N so that M and N are oriented. We say that f is *orientation preserving* if there exists a strictly positive real valued smooth function λ on M such that $f^*\Xi = \lambda\Theta$.

Notice that on a given orientable manifold M there are different possible choices of nowhere null d -forms that induce the same orientation. It turns out that there

are exactly two classes of such forms, each one comprised by all the nowhere null d -forms that differ for a strictly positive factor $\lambda \in C^\infty(M)$, such that each element of a class induce the same orientation on M . Usually an oriented manifold M is denoted by (M, \mathfrak{o}) , where \mathfrak{o} is one of the above mentioned classes of nowhere null d -forms.

Once that an orientation \mathfrak{o} on M is chosen, for each point $p \in M$ it is possible to find a base $\{v_1, \dots, v_d\}$ of $T_p M$ such that $\Omega(p)(v_1, \dots, v_d) > 0$ for each $\Omega \in \mathfrak{o}$. We say that such base is *oriented*. If M is also endowed with a metric g , g_p defines an inner product on the vector space $T_p M$ for each $p \in M$. This allows us to choose g_p -orthonormal bases of $T_p M$ for any point p in M . The next theorem puts together the choice of an orientation \mathfrak{o} and the presence of a metric to provide a univocal way to choose a d -form in \mathfrak{o} .

Theorem 1.1.37. *Let (M, \mathfrak{o}) be an oriented d -dimensional manifold endowed with a metric g . Then there exists a unique nowhere null d -form $d\mu_g \in \mathfrak{o}$, called volume form over (M, \mathfrak{o}) induced by g , such that for each $p \in M$ $d\mu_g$ takes the value $+1$ on each oriented orthonormal base of $T_p M$.*

It turns out that for each local coordinate neighborhood (U, Ω, ϕ) of M the following equation holds on every point of Ω :

$$\phi_*(d\mu_g) = \sqrt{|\det g|} dx^1 \wedge \dots \wedge dx^d = \sqrt{|\det g|} dV, \quad (1.1.4)$$

where $\{dx^1, \dots, dx^d\}$ is the base of $T^*\Omega$ (identified with $\Omega \times \mathbb{R}^d$) defined by $dx^i(x) = e_i$ for each $i \in \{1, \dots, d\}$ and each $x \in \Omega$, where $\{e_1, \dots, e_d\}$ is an oriented orthonormal base of \mathbb{R}^d endowed with a (non necessarily positive definite) inner product with the same signature of g .

The volume form $d\mu_g$ provided by the last theorem will become very useful in the next subsection when we will introduce a notion of integral on a manifold. An example of volume form is the standard measure $dV = dx^1 \wedge \dots \wedge dx^d$ of \mathbb{R}^d that appears in eq. (1.1.4) above.

Before we proceed with the next subsection, we want to introduce two new operators. The first one is the Hodge dual. The detailed procedure used to define it can be found in [Jos95, Sect. 2.1, pp. 87-90].

Consider an oriented d -dimensional manifold (M, \mathfrak{o}) endowed with a metric g . Let $d\mu_g \in \mathfrak{o}$ be the volume form over (M, \mathfrak{o}) induced by g . Since g defines a non degenerate inner product on each cotangent space $T_p^* M$, we can use it, together with the volume form, to choose an orthonormal base $\{\omega_1^{(p)}, \dots, \omega_d^{(p)}\}$ of $T_p^* M$ for each $p \in M$ such that $d\mu_g(p)(\omega_1^{(p)}, \dots, \omega_d^{(p)}) = +1$. Notice that a base of $\Lambda_p^k M$ is provided by

$$B_{p,k} = \left\{ \omega_{i_1}^{(p)} \wedge \dots \wedge \omega_{i_k}^{(p)} : 1 \leq i_1 < \dots < i_k \leq d \right\}.$$

We are ready to define the Hodge dual.

Definition 1.1.38. Let (M, \mathfrak{o}) be a d -dimensional oriented manifold endowed with a metric g and let $d\mu_g \in \mathfrak{o}$ be the volume form over (M, \mathfrak{o}) induced by g . For each $p \in M$ and each $k \in \{1, \dots, d\}$, we define the *Hodge dual* $*$ as the unique linear map from $\Lambda_p^k M$ to $\Lambda_p^{d-k} M$ satisfying the following condition for each element of $B_{p,k}$:

$$* \left(\omega_{i_1}^{(p)} \wedge \dots \wedge \omega_{i_k}^{(p)} \right) = \omega_{j_1}^{(p)} \wedge \dots \wedge \omega_{j_{d-k}}^{(p)},$$

where $j_1, \dots, j_{d-k} \in \{1, \dots, d\}$ are chosen in such a way that

$$\left\{ \omega_{i_1}^{(p)}, \dots, \omega_{i_k}^{(p)}, \omega_{j_1}^{(p)}, \dots, \omega_{j_{d-k}}^{(p)} \right\}$$

is an oriented base of $T_p^* M$.

It can be shown that this definition is well posed so that for each $p \in M$ and each $k \in \{1, \dots, d\}$ we have at our disposal the operator $*$. If we consider $\Theta \in \Omega^k M$, we can take $*(\Theta(p))$ for each $p \in M$. It turns out that the map

$$\begin{aligned} M &\rightarrow \Lambda^{d-k} M \\ p &\mapsto *(\Theta(p)) \end{aligned}$$

is a smooth section in $\Lambda^{d-k} M$ that we denote with $*\Theta$. Then the Hodge dual naturally defines an operator $*$ from $\Omega^k M$ to $\Omega^{d-k} M$. This can be done for each $k \in \{1, \dots, d\}$ so that the Hodge dual is defined as an operator on ΩM .

Remark 1.1.39. From the last definition it is possible to deduce a formula for the components of the Hodge dual of a k -form. Let $\omega \in \Omega^k M$ and consider a point $p \in M$ and a coordinate neighborhood (U, V, ϕ) of p in M . On $T_p^* M = \Lambda_p^1 M$ we choose the oriented orthonormal basis $\{dx^1, \dots, dx^d\}$ and we denote the totally antisymmetric symbol with $\varepsilon_{i_1 \dots i_d}$. Then the components of $*\omega$ in p in the basis of $\Lambda_p^{d-k} M$ are given by the formula

$$(*\omega)_{i_1 \dots i_{d-k}} = \frac{1}{k!} \omega_{j_1 \dots j_k} g^{j_1 j'_1} \dots g^{j_k j'_k} \varepsilon_{j'_1 \dots j'_k, i_1 \dots i_{d-k}} \sqrt{|\det g|},$$

where $\omega_{j_1 \dots j_k}$ are the components of ω at p in the basis of $\Lambda_p^k M$, i.e.

$$\omega(p) = \frac{1}{k!} \omega_{j_1 \dots j_k} dx^{j_1} \wedge \dots \wedge dx^{j_k},$$

and (g^{ij}) is the inverse of the matrix (g_{ij}) , whose coefficients are given by

$$g_{ij} = g_p \left((dx^i)^\sharp, (dx^j)^\sharp \right).$$

There are some other very important properties of the Hodge dual. We recollect them in the following theorem.

Proposition 1.1.40. *Let (M, \mathfrak{o}) be a d -dimensional oriented manifold endowed with a metric g with signature $s = \pm 1$. The Hodge dual $*$: $\Lambda M \rightarrow \Lambda M$ satisfies the following properties:*

- for each $p \in M$, each $k \in \{1, \dots, d\}$ and each $\omega \in \Lambda_p^k M$ it holds that

$$**\omega = s(-1)^{k(d-k)}\omega;$$

- for each $p \in M$, each $k \in \{1, \dots, d\}$ and each $\omega, \theta \in \Lambda_p^k M$ it holds that

$$*(\omega \wedge *\theta) = s \langle \omega, \theta \rangle_{g,k},$$

where $\langle \cdot, \cdot \rangle_{g,k}$ denotes the inner product of the vector bundle $\Lambda^k M$ induced by the metric g .

Moreover, if (N, \mathfrak{p}) is an oriented manifold endowed with a metric h with signature $s' = s$ and f is an orientation preserving embedding such that $g = f^*h$, we have that

$$f^* \circ *^N = *^M \circ f^*.$$

The last theorem has three very important consequences:

- For each $p \in M$ and each $k \in \{1, \dots, d\}$ $*$: $\Lambda_p^k M \rightarrow \Lambda_p^{d-k} M$ is a vector space isomorphism whose inverse $*^{-1} = s(-1)^{k(d-k)}*$ can be extended to $\Lambda^{d-k} M$ because it coincides with $*$: $\Lambda_p^{d-k} M \rightarrow \Lambda_p^k M$ (up to a ± 1 factor). Hence $*$: $\Lambda^k M \rightarrow \Lambda^{d-k} M$ is a vector bundle isomorphism.
- It is easy to show that $*1 = d\mu_g(p)$, where 1 is in $\Lambda_p^0 M = \mathbb{R}$.
- We can use the wedge product and the Hodge dual to completely characterize the inner product $\langle \cdot, \cdot \rangle_{g,k}$ induced by the metric g on $\Lambda^k M$ and we note that the section $\langle \omega, \theta \rangle_{g,k} \in C^\infty(M)$ coincides with the section $s * (\omega \wedge *\theta)$ for each $\omega, \theta \in \Omega^k M$.

As anticipated, we conclude this subsection with the introduction of the codifferential.

Definition 1.1.41. Let (M, \mathfrak{o}) be a d -dimensional oriented manifold endowed with a metric g with signature $s = \pm 1$. For each $k \in \{1, \dots, d\}$ we call *codifferential* the map $\delta : \Omega^k M \rightarrow \Omega^{k-1} M$ defined by

$$\delta = (-1)^k *^{-1} \circ d \circ * = s(-1)^{dk+d+1} * \circ d \circ *.$$

We say that a k -form Θ is *coclosed* when $\delta\Theta = 0$.

Notice that, as a consequence of the property $d^2 = 0$, it follows also that $\delta^2 = 0$. Moreover, if f is an orientation preserving embedding from the oriented manifold (M, \mathfrak{o}) to the oriented manifold (N, \mathfrak{p}) and if M is endowed with a metric g of signature s , while N is endowed with a metric h with signature $s' = s$ such that $g = f^*h$, then it holds that

$$f^* \circ \delta_N = \delta_M \circ f^*.$$

1.1.4 Integration on a manifold

In the last subsection we discussed some questions about the calculus of differential forms. In particular, considering a d -dimensional manifold M , we used the space $\Omega^d M$ of d -forms over M to introduce the orientability of a manifold. This concept allows us to define a notion of integral on a manifold. The precise procedure to define the integral on a manifold is shown in detail, for example, in [Boo86, Chap. VI]. Here we briefly present such construction restricting to smooth functions.

Suppose that M is an orientable d -dimensional manifold and that we have chosen a nowhere null d -form Θ that defines an orientation \mathfrak{o} on M so that (M, \mathfrak{o}) becomes an oriented manifold. It is possible to express any other d -form Ξ over M as a product $f\Theta$, where $f \in C^\infty(M)$. We say that a function of $C^\infty(M)$ is integrable if it has compact support, i.e. if it belongs to $\mathcal{D}(M)$, and moreover a d -form over M is said to be *integrable* if it can be expressed as a product $f\Theta$, with an integrable function f . This definition of integrable d -form does not depend on the choice of the particular d -form Θ used to define the orientation \mathfrak{o} on M . Notice that in our simplified treatment the set of integrable d -forms coincides exactly with the space of d -forms with compact support, denoted by $\Omega_0^d M$.

The *integral of an integrable d -form* is defined in first place on a particular subset of $\Omega_0^d M$ constituted by those d -forms Ξ whose support is contained in some coordinate neighborhood (U, V, ϕ) : using the local coordinates, we write

$$\phi_* \Xi|_U = h(x) dx^1 \wedge \cdots \wedge dx^d \quad \forall x \in V,$$

where $h \in \mathcal{D}(V)$, and then we set

$$\int_M \Xi = \int_V h(x) dV,$$

where dV is the standard measure on \mathbb{R}^d . It can be shown that this definition is independent of the choice of (U, V, ϕ) (provided that only coordinate neighborhoods having transition charts with positive Jacobian determinant are considered). In second place such definition is extended to any integrable d -form Ξ with the help of a particular partition of unity that reduces Ξ to a finite sum of d -forms of the type considered in first place. Again it is possible to prove that this definition does not

depend on the particular choices made.

The next theorem recollects some properties of the integral.

Theorem 1.1.42. *Let M be an orientable d -dimensional manifold. Let Θ be a d -form over M defining an orientation \mathfrak{o} on M . The construction above defines the integral of integrable d -forms over M , specifically the map $\Xi \in \Omega_0^d M \mapsto \int_M \Xi \in \mathbb{R}$. Such map fulfils the following properties:*

- \mathbb{R} -linearity: for each $a, b \in \mathbb{R}$ and each $\Xi, \Xi' \in \Omega_0^d M$ it holds that

$$\int_M (a\Xi + b\Xi') = a \int_M \Xi + b \int_M \Xi';$$

- if $\Xi \in \Omega_0^d M$ can be expressed as $h\Theta$ with some non negative real valued smooth function h , we have $\int_M \Xi \geq 0$ and $\int_M \Xi = 0$ if and only if $h = 0$;
- if $f : M \rightarrow N$ is an orientation preserving embedding between the oriented d -dimensional manifolds (M, \mathfrak{o}) and (N, \mathfrak{p}) , the following equation holds for each $\Xi \in \Omega_0^d N$:

$$\int_M f^* \Xi = \int_N \Xi.$$

Till now we considered only the integration on an orientable d -dimensional manifold M of d -forms. However we would like to integrate also functions of $\mathcal{D}(M)$ as in the case of ordinary integrals on Euclidean spaces. In the general case this cannot be done because a measure on an arbitrary orientable manifold is missing. As a matter of fact, once that an orientation \mathfrak{o} on M is chosen, each $\Omega \in \mathfrak{o}$ provides a possible measure on M and it is not possible for us to make a particular choice that reduces to the standard measure dV when $M = \mathbb{R}^d$. Nevertheless, when M is endowed with a metric g and an orientation \mathfrak{o} has been chosen, we are able to pick out the volume form $d\mu_g \in \mathfrak{o}$ exploiting Theorem 1.1.37. Using $d\mu_g$ we are able to evaluate in an unambiguous way the integrals of functions in $\mathcal{D}(M)$. Moreover it can be shown that, when M is an open subset of the vector space \mathbb{R}^d endowed with the usual inner product of Euclidean spaces as metric, $d\mu_g$ reduces to the ordinary measure dV .

In the development of the thesis we will make extensive use of Stokes' theorem on manifolds. Before we are ready to present its statement, we must introduce a slight extension of the notion of manifold. This extension requires the introduction of the half plane, i.e. the following subset of \mathbb{R}^d :

$$H^d = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_d \geq 0\}.$$

We also take the chance to define the boundary of H^d as

$$\partial H^d = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_d = 0\}.$$

Definition 1.1.43. A d -dimensional manifold (with boundary) M is a connected Hausdorff topological space with a countable basis of open subsets such that for each point $p \in M$ there exists a triple (U, Ω, ϕ) , called *coordinate neighborhood* (or *local chart*), where U is an open neighborhood of p in M , Ω is an open subset of H^d and $\phi : U \rightarrow \Omega$ is a homeomorphism. Moreover there are other two requirements:

- there exists a (*smooth*) *atlas*, which is a collection $\{(U_\alpha, \Omega_\alpha, \phi_\alpha)\}_{\alpha \in I}$ of coordinate neighborhoods in M , where I is an index set, such that $\{U_\alpha\}_{\alpha \in I}$ is an open covering of M and the map, called *transition chart*,

$$\begin{aligned} T_{\phi_\alpha}^{\phi_\beta} : \Omega_\alpha \cap \Omega_\beta &\rightarrow \Omega_\alpha \cap \Omega_\beta \\ x &\mapsto (\phi_\beta \circ \phi_\alpha^{-1})(x) \end{aligned}$$

is a diffeomorphism for each $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$;

- there exists a *maximal atlas*, i.e. an atlas that contains each coordinate neighborhood (U, Ω, ϕ) such that the transition maps $T_{\phi_\alpha}^\phi$ and $T_\phi^{\phi_\alpha}$ are diffeomorphisms for each $\alpha \in I$ with $U_\alpha \cap U \neq \emptyset$.

For a detailed discussion about manifolds with boundary the reader is referred to [Boo86, Chap. VI, Sect. 4]

For a d -dimensional manifold with boundary M , it makes sense to define a subset ∂M , called *boundary of M* . Such subset consists of the points of M that are preimages of points of ∂H^d through some coordinate neighborhood. It turns out that ∂M is a $(d - 1)$ -dimensional manifold with topology and differentiable structure induced by those of M and that the inclusion map $\iota : \partial M \rightarrow M$ is an embedding. Notice that $M \setminus \partial M$ is a manifold in the ordinary sense and that M itself is actually a manifold in the ordinary sense if ∂M is empty.

Everything we said till this point about manifolds can be extended to manifolds with boundary in an almost straightforward way. The only situation in which it is possible to face some troubles is the definition of the tangent space at a point of the boundary. A possible approach to such problem is presented in [Boo86, p. 254]. Once that such problem is overcome we can indeed define differential forms on these new type of manifolds.

Suppose we are dealing with an oriented manifold with boundary and we want to make an integral on its boundary submanifold. In order to give sense to integrals on the boundary we need a notion of orientability of the boundary and the choice of a specific orientation. The following theorems answers to our question.

Theorem 1.1.44. *Let (M, \mathfrak{o}) be an oriented manifold with non empty boundary ∂M . Then ∂M is itself an orientable manifold and the orientation \mathfrak{o} of M determines uniquely an orientation \mathfrak{o}' on ∂M .*

Consider an oriented d -dimensional manifold (M, \mathfrak{o}) , a point $p \in \partial M$ and a coordinate neighborhood (U, Ω, ϕ) of p . We have that $\phi(p) \in \partial H^d$ and that each $v \in T_p M$ can be classified as *inward pointing*, *outward pointing* or *tangent to ∂M* if its last component in the basis induced by the chosen coordinate neighborhood is respectively positive, negative or null. Such classification turns out to be independent of the particular coordinate neighborhood and of the orientation of M . The orientation \mathfrak{o}' provided by the last theorem can be characterized in the following way: if p is a point of the boundary ∂M and $v \in T_p M$ is outward pointing, a base $\{v_1, \dots, v_{d-1}\}$ of $T_p \partial M$ is oriented if and only if $\{v_1, \dots, v_{d-1}, v\}$ is an oriented base of $T_p M$.

If (M, \mathfrak{o}) is an oriented d -dimensional manifold with non empty boundary ∂M and M is endowed with a metric g , we immediately have an orientation \mathfrak{o}' on ∂M provided by the last theorem and a metric g' obtained via pull-back of g through the inclusion map ι of ∂M into M (remember that $\iota_{\partial M}^M$ is actually an embedding). Then we have a volume form on the boundary ∂M , that we denote by dS_g , that provides a precise notion of integral on the boundary.

We are now able to state Stokes' theorem on an arbitrary manifold with boundary. A thorough discussion about this topic can be found, for example, in [Boo86, Chap. VI, Sect. 5].

Theorem 1.1.45. *Let (M, \mathfrak{o}) be an oriented d -dimensional manifold with (eventually empty) boundary ∂M , let \mathfrak{o}' denote the orientation of ∂M determined by \mathfrak{o} and let $\iota : \partial M \rightarrow M$ be the inclusion map (actually an embedding). For each $\Xi \in \Omega_0^{d-1} M$ we have that*

$$\int_M d\Xi = \int_{\partial M} \iota^* \Xi.$$

Notice that, if ∂M is empty, then the RHS is always null. For us this will always be the case since we will always consider manifolds as defined in Definition 1.1.1, which is to say manifolds with empty boundary.

With the help of the Hodge dual, defined in the previous subsection, we can introduce an inner product between k -forms with compact support on an oriented manifold endowed with a metric. With such notion we can prove that the codifferential δ is formally adjoint to the exterior derivative d .

Proposition 1.1.46. *Let (M, \mathfrak{o}) be an oriented d -dimensional manifold endowed with a metric g and let $*$ be the Hodge dual. For each $k \in \{1, \dots, d\}$ consider the set*

$$S^k = \{(\Xi, \Xi') \in \Omega^k M \times \Omega^k M : \text{supp}(\omega) \cap \text{supp}(\theta) \text{ is a compact subset of } M\}.$$

We have that the map

$$\begin{aligned} (\cdot, \cdot)_{g,k} : S^k &\rightarrow \mathbb{R} \\ (\Xi, \Xi') &\mapsto \int_M (\Xi \wedge * \Xi') \end{aligned}$$

defines a non degenerate inner product on the vector space $\Omega_0^k M$.

The integrand $\Xi \wedge * \Xi'$ in the definition above may be rewritten as

$$\Xi \wedge * \Xi' = *^{-1} * (\Xi \wedge * \Xi') = \langle \Xi, \Xi' \rangle_{g,k} d\mu_g.$$

Then we can express $\langle \Xi, \Xi' \rangle_{g,k}$ for $\Xi, \Xi' \in \Omega^k M$ as the integral of the section $\langle \Xi, \Xi' \rangle_{g,k} \in \Omega^0 M$:

$$\int_M (\Xi \wedge * \Xi') = \int_M \langle \Xi, \Xi' \rangle_{g,k} d\mu_g.$$

Notice that, if g is a Riemannian metric, then $(\cdot, \cdot)_{g,k}$ is even a scalar product and so $(\Omega_0^k M, (\cdot, \cdot)_g)$ is a pre-Hilbert space.

As announced $(\cdot, \cdot)_{g,k}$ allows us to establish a particular relation between the exterior derivative and the codifferential. Such a relation is a direct consequence of Stokes' theorem.

Proposition 1.1.47. *Let (M, \mathfrak{o}) be an oriented d -dimensional manifold with empty boundary endowed with a metric g with signature $s = \pm 1$. Then the codifferential δ is formally adjoint to the exterior derivative d , i.e. for each $k \in \{1, \dots, d\}$, each $\Xi \in \Omega_0^{k-1} M$ and each $\Xi' \in \Omega_0^k M$ the following equation holds:*

$$(\Xi, \delta \Xi')_{g,k-1} = (d\Xi, \Xi')_{g,k}.$$

Proof. Fix $k \in \{1, \dots, d\}$, $\Xi \in \Omega_0^{k-1} M$ and $\Xi' \in \Omega_0^k M$. Since M has empty boundary, Stokes' theorem implies that

$$\int_M d(\Xi \wedge * \Xi') = 0.$$

On the other hand we have:

$$d(\Xi \wedge * \Xi') = d\Xi \wedge * \Xi' + (-1)^{k-1} \Xi \wedge d * \Xi'.$$

Recalling the definition of the codifferential (cfr. Definition 1.1.41), we see that

$$d * \Xi' = * *^{-1} d * \Xi' = (-1)^k * \delta \Xi'.$$

Hence we deduce that

$$d(\Xi \wedge * \Xi') = d\Xi \wedge * \Xi' - \Xi \wedge * d\Xi',$$

from which it follows

$$0 = \int_M d(\Xi \wedge * \Xi') = \int_M (d\Xi \wedge * \Xi') - \int_M (\Xi \wedge * d\Xi').$$

Then the definition of $(\cdot, \cdot)_{g,k}$ allows us to conclude the proof. \square

1.2 Lorentzian geometry

This section is devoted to the presentation of some notions concerning Lorentzian geometry and in particular global hyperbolicity. The interested reader should refer to [O'N83] for a deeper insight in this subject.

1.2.1 Lorentzian manifolds

Definition 1.2.1. We call *Lorentzian manifold* a pair (M, g) where M is an orientable d -dimensional manifold and g is a Lorentzian metric on M .

Notice that we have included the requirement of orientability in the definition of Lorentzian manifold. This is indeed not necessary if one wants to study Lorentzian manifolds in general, however in the development of this thesis we will often make use of Stokes' theorem, which requires the orientability of the manifold to hold.

We take the chance to introduce some notions which prove to be very helpful in the discussion of the causal structure of a Lorentzian manifold.

Definition 1.2.2. Consider a Lorentzian manifold (M, g) . For each point $p \in M$ and each tangent vector $v \in T_p M$ we say that v is

- *g -timelike* if $g_p(v, v) < 0$,
- *g -lightlike* if $g_p(v, v) = 0$,
- *g -causal* if $g_p(v, v) \leq 0$,
- *g -spacelike* if $g_p(v, v) > 0$.

Note that, if there is no risk of misunderstanding (e.g. when we consider only one metric on a specified manifold), we often do not make explicit the metric so that, for example, we simply speak of timelike tangent vectors, instead of g -timelike tangent vectors. Anyway in this section such omission is not adopted in order to underline the dependence on the metric of the objects that we define.

With the definitions given above we can define a new property of Lorentzian manifolds, called time orientability. This concept is associated to the idea of finding some “preferred direction” on our manifold that can be interpreted as a direction of “time progress” in accordance with the given metric.

Definition 1.2.3. We say that a Lorentzian manifold (M, g) is *time orientable* if there exists a vector field $T \in C^\infty(M, TM)$ over M such that $T(p)$ is g -timelike for each $p \in M$. We call *time orientation* of the time orientable Lorentzian manifold (M, g) each one of the connected components of the set of everywhere g -timelike vector fields over M .

Then we call *oriented and time oriented Lorentzian manifold* a quadruple $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ where

- (M, g) is a time orientable Lorentzian manifold,
- \mathfrak{o} is an orientation on M ,
- \mathfrak{t} is a time orientation on (M, g) .

With the last two definition we are able to introduce a classification of the curves in an oriented and time oriented Lorentzian manifold \mathcal{M} .

Definition 1.2.4. Consider a Lorentzian manifold (M, g) and a C^1 curve $\gamma : I \rightarrow M$, where $I \subseteq \mathbb{R}$ is an interval.

We define the vector $\dot{\gamma}(p)$ *tangent to the curve γ in the point p along the curve* in the following way: If $t \in I$ such that $\gamma(t) = p$, we consider the curve $\gamma_t(s) = \gamma(s + t)$ defined for s in a sufficiently small interval containing 0 and we set $\dot{\gamma}(p) = [\gamma_t(s)]$ (for the meaning of $[\cdot]$ see Definition 1.1.3).

We say that γ is *g -timelike*, *g -lightlike*, *g -causal* or *g -spacelike* if $\dot{\gamma}(p)$ is such for each p along γ .

If (M, g) is time orientable, \mathfrak{o} is an orientation of M and \mathfrak{t} is a time orientation of (M, g) (so that $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ is an oriented and time oriented Lorentzian manifold) and if the curve γ is g -causal we say that it is:

- *\mathfrak{t} -future directed* if $g_p(\dot{\gamma}(p), \mathfrak{t}(p)) < 0$ for each p along γ ,
- *\mathfrak{t} -past directed* if $g_p(\dot{\gamma}(p), \mathfrak{t}(p)) > 0$ for each p along γ .

This definition extends to piecewise C^1 curves considering separately each C^1 piece.

It may happen that we omit the explicit indication of the metric and the time orientation chosen on the manifold when we deal with curves. Clearly such omission will be done only if there is no possibility of misunderstanding. For example, when we deal with an oriented and time oriented Lorentzian manifold $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ and there is no other oriented and time oriented Lorentzian manifold with the same underlying manifold M but with different metric or time orientation, you may find the

expression “future directed timelike curve”, instead of “ \mathfrak{t} -future directed g -timelike curve”.

Now we define some particular subsets of M . These subsets, as we will see, are very helpful in the characterization of the causal structure of $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$.

Definition 1.2.5. Consider an oriented and time oriented Lorentzian manifold $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, a subset $S \subseteq M$ and a point $p \in S$. We define:

- the \mathcal{M} -chronological future of the point p in S , denoted by $I_+^{\mathcal{M},S}(p)$, as the subset of S constituted by the points $q \in S \setminus \{p\}$ such that there exists a \mathfrak{t} -future directed g -timelike curve starting from p and ending in q which is entirely contained in S ;
- the \mathcal{M} -causal future of the point p in S , denoted by $J_+^{\mathcal{M},S}(p)$, as the subset of S constituted by p and the points $q \in S$ such that there exists a \mathfrak{t} -future directed g -causal curve starting from p and ending in q which is entirely contained in S .

We also define the \mathcal{M} -chronological past of the point p in S , denoted by $I_-^{\mathcal{M},S}(p)$, and the \mathcal{M} -causal past of the point p in S , denoted by $J_-^{\mathcal{M},S}(p)$, with the substitution of the word “future” with the word “past” in the definitions of $I_+^{\mathcal{M},S}(p)$ and $J_+^{\mathcal{M},S}(p)$.

We extend the definitions of these subsets from arbitrary points $p \in M$ to arbitrary subsets $\Omega \subseteq S$ taking the union over the points in Ω , e.g. we define the \mathcal{M} -chronological future of the subset Ω in S as $I_+^{\mathcal{M},S}(\Omega) = \bigcup_{p \in \Omega} I_+^{\mathcal{M},S}(p)$, and we denote the unions $I_+^{\mathcal{M},S}(\Omega) \cup I_-^{\mathcal{M},S}(p)$ and $J_+^{\mathcal{M},S}(p) \cup J_-^{\mathcal{M},S}(p)$ with $I^{\mathcal{M},S}(p)$ and respectively with $J^{\mathcal{M},S}(p)$.

Finally we define the *Cauchy development* of S in \mathcal{M} as the subset $D^{\mathcal{M}}(S)$ comprised by the points $q \in M$ such that every inextendible \mathfrak{t} -future directed (or equivalently \mathfrak{t} -past directed) g -causal curve in M passing through q meets S .

We invite the reader to bear in mind that, when there is no risk of ambiguity, we may write $I_+^S(p)$ in place of $I_+^{\mathcal{M},S}(p)$. Moreover in our notation we always omit the subset S in when S is the entire manifold so that we write $I_+^{\mathcal{M}}(p)$ in place of $I_+^{\mathcal{M},M}(p)$.

The notions of causal future and causal past allow us to define future compact and past compact subsets.

Definition 1.2.6. Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be an oriented and time oriented Lorentzian manifold and let S be a subset of M . We say that S is

- \mathcal{M} -past compact if $S \cap J_-^{\mathcal{M}}(p)$ is compact for each $p \in M$;
- \mathcal{M} -future compact if $S \cap J_+^{\mathcal{M}}(p)$ is compact for each $p \in M$.

Past compact and future compact subsets will play an important role in the next section, when we will study the properties of Green operators.

When dealing with Lorentzian manifolds, we can establish the notion of causal separation. We will use such notion when we will introduce the generally covariant locality principle in Chapter 2.

Definition 1.2.7. Let (M, g) be a Lorentzian manifold. We say that two subsets S_1 and S_2 of M are (M, g) -causally separated (or simply *causally separated*, when there is no risk of misunderstanding) if there is no g -causal curve on M that connects a point of S_1 and a point of S_2 .

Sometimes we may say that S_1 is (M, g) -causally separated from S_2 , meaning that there is no point of S_1 that can be connected through some causal curve to a point of S_2 . Obviously this is equivalent to saying that S_1 and S_2 are (M, g) -causally separated.

Remark 1.2.8. We can give a condition that is equivalent to causal separation on an oriented and time oriented Lorentzian manifold \mathcal{M} . We can show that S_1 and S_2 are \mathcal{M} -causally separated if and only if $J^\mathcal{M}(S_1) \cap S_2 = \emptyset$ (or equivalently $J^\mathcal{M}(S_2) \cap S_1 = \emptyset$). This follows from the fact that the points of $J^\mathcal{M}(S_1)$ are by definition connected to points of S_1 through some g -causal curve in M . Hence the intersection $J^\mathcal{M}(S_1) \cap S_2$ consists exactly of those points of S_2 that are connected to points of S_1 through some g -causal curve in M . Then $J^\mathcal{M}(S_1) \cap S_2 = \emptyset$ means that there are not points of S_2 that are connected to points of S_1 through some g -causal curve in M , i.e. S_2 is \mathcal{M} -causally separated from S_1 or, equivalently, S_1 and S_2 are \mathcal{M} -causally separated.

Now we introduce the notion of causal compatibility and the notion of causal convexity. Loosely speaking causal compatibility means that the causal future (or past) of a point in a subset S of an oriented and time oriented Lorentzian manifold coincides with the intersection with S of the causal future (or past) of such point taken in the whole manifold. Instead causal convexity requires that each pair of points in a subset can be connected by a causal curve contained in such subset.

Definition 1.2.9. Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be an oriented and time oriented Lorentzian manifold and let S be a subset of M . We say that S is

- \mathcal{M} -causally compatible if $J_{\pm}^{\mathcal{M}, S}(p) = J_{\pm}^{\mathcal{M}}(p) \cap S$ for each $p \in S$;
- \mathcal{M} -causally convex if each \mathfrak{t} -future (or equivalently \mathfrak{t} -past) directed g -causal curve in M that starts and ends in S is entirely contained in S .

We observe that, since each \mathfrak{t} -future/past directed g -causal curve that is contained in $S \subseteq M$ can be directly seen also as a \mathfrak{t} -future/past directed g -causal curve contained in M , it always holds the inclusion $J_{\pm}^{\mathcal{M}, S}(p) \subseteq J_{\pm}^{\mathcal{M}}(p) \cap S$ for each $p \in S$. Hence the real condition of causal compatibility is the other inclusion.

Remark 1.2.10. It is easily seen that causal convexity implies causal compatibility. Suppose that S is an \mathcal{M} -causally convex subset of M . Once that a point $p \in S$ is fixed, we can consider a point q in $J_+^{\mathcal{M}}(p) \cap S$ (or in $J_-^{\mathcal{M}}(p) \cap S$). Because of causal convexity each \mathfrak{t} -future (or respectively \mathfrak{t} -past directed) g -causal curve in M from p to q is entirely contained in S . From the definition of causal future (respectively causal past) at least one such curve exists and this implies that q falls in $J_{\pm}^{\mathcal{M},S}(p)$ as required by causal compatibility.

Remark 1.2.11. Each causally compatible connected open subset of an oriented and time oriented Lorentzian manifold can be interpreted as an oriented and time oriented Lorentzian manifold in its own right. For example take the d -dimensional oriented and time oriented Lorentzian manifold $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ and let Ω be a causally compatible connected open subset of M . Then undoubtedly Ω can be seen as a d -dimensional submanifold of M (cfr. Remark 1.1.7) and hence a d -dimensional manifold in its own right. Moreover it becomes an oriented and time oriented Lorentzian manifold when endowed with $g|_{\Omega}$, $\mathfrak{o}|_{\Omega}$ and $\mathfrak{t}|_{\Omega}$, where for $\mathfrak{o}|_{\Omega}$ we mean the class of nowhere null d -forms over Ω that includes the restrictions to Ω of the nowhere null d -forms over M contained in \mathfrak{o} . We denote the oriented and time oriented Lorentzian manifold $(\Omega, g|_{\Omega}, \mathfrak{o}|_{\Omega}, \mathfrak{t}|_{\Omega})$ with $\mathcal{M}|_{\Omega}$.

Notice that $J_{\pm}^{\mathcal{M}|_{\Omega}}(p) = J_{\pm}^{\mathcal{M},\Omega}(p)$ for each $p \in \Omega$ as a direct consequence of the definition of causal future/past.

Since causal convexity implies causal compatibility, the same conclusions hold also for causally convex subsets of oriented and time oriented Lorentzian manifolds.

1.2.2 Globally hyperbolic spacetimes

Now we present the notion of global hyperbolicity. Such concept is the key hypothesis for a theorem that states existence and uniqueness of global solutions for a wave equation with proper initial data on an oriented and time oriented Lorentzian manifold. Hence global hyperbolicity will be an unavoidable request throughout the rest of the thesis.

Definition 1.2.12. Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be an oriented and time oriented Lorentzian manifold. A subset $S \subseteq M$ is said to be \mathcal{M} -globally hyperbolic if the following conditions hold:

- S fulfils the *g-causality condition*, i.e. there are no closed g -causal curves in S ;
- $J_+^{\mathcal{M},S}(p) \cap J_-^{\mathcal{M},S}(q)$ is a compact subset of S for each $p, q \in S$ with respect to the topology naturally induced on S by the topology of M .

We call *globally hyperbolic spacetime* each oriented and time oriented Lorentzian manifold $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ such that M is a \mathcal{M} -globally hyperbolic subset.

Originally global hyperbolicity of (oriented and) time oriented Lorentzian manifolds required a stricter condition than that of causality, which is called *strong causality condition*. Such condition requires that there are no “almost closed” g -causal curves. A precise statement of the strong causality condition on an (oriented and) time oriented Lorentzian manifold $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ is the following: For each point $p \in M$ and for each open neighborhood U of p in M there exists an open neighborhood $V \subseteq U$ of p in M such that each \mathfrak{t} -future directed (or equivalently \mathfrak{t} -past directed) g -causal curve which starts and ends in V must be entirely contained in U . However this stricter requirement is equivalent to the causality condition in the present context as was shown by Bernal and Sanchez in [BS07].

Remark 1.2.13. Globally hyperbolic connected open subsets of oriented and time oriented Lorentzian manifolds can be considered as globally hyperbolic spacetimes in their own right. To see this, consider an oriented and time oriented Lorentzian manifold $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ and a \mathcal{M} -globally hyperbolic open subset Ω of M . From Remark 1.2.11 we have that $\mathcal{M}|_\Omega$ is itself an oriented and time oriented Lorentzian manifold and that $J_\pm^{\mathcal{M}|_\Omega}(p) = J_\pm^{\mathcal{M}, \Omega}(p)$ for each $p \in \Omega$. Since per hypothesis Ω is \mathcal{M} -globally hyperbolic, it follows also that Ω is $\mathcal{M}|_\Omega$ -globally hyperbolic too. Therefore $\mathcal{M}|_\Omega$ is itself a globally hyperbolic spacetime.

Definition 1.2.14. Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be an oriented and time oriented Lorentzian manifold and let S be a subset of M . S is *achronal in \mathcal{M}* if it is met at most once by each \mathfrak{t} -future directed (or equivalently \mathfrak{t} -past directed) g -timelike curve in M . S is *acausal in \mathcal{M}* if it is met at most once by each \mathfrak{t} -future directed (or equivalently \mathfrak{t} -past directed) g -causal curve in M .

We say that Σ is a *Cauchy surface of \mathcal{M}* if each inextendible \mathfrak{t} -future directed (or equivalently \mathfrak{t} -past directed) g -timelike curve in M passing through p meets Σ exactly once.

Obviously each acausal subset is also achronal and each Cauchy surface is achronal. It can be proved that a Cauchy surface Σ is a closed achronal topological hypersurface met by each inextendible causal curve at least once [O’N83, Chap. 14, Lem. 29, p. 415], hence its Cauchy development $D^{\mathcal{M}}(\Sigma)$ coincides with M .

With the last definition we have at our disposal all the material needed to state a very important theorem that provides two handy conditions that are equivalent to global hyperbolicity.

Theorem 1.2.15. *Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be an oriented and time oriented Lorentzian manifold. Then the following conditions are equivalent:*

- \mathcal{M} is globally hyperbolic;
- there exists a Cauchy surface Σ of \mathcal{M} ;

- *there exists a diffeomorphism from the manifold M to the manifold $\mathbb{R} \times \Sigma$, where Σ is a $(d - 1)$ -dimensional manifold, such that the push-forward of the metric g through such diffeomorphism takes the form $-\beta dt^2 + g_t$, where β is a smooth strictly positive function of $t \in \mathbb{R}$, g_t is a Riemannian metric on $\{t\} \times \Sigma$ for each $t \in \mathbb{R}$ and the family of Riemannian metrics $\{g_t, t \in \mathbb{R}\}$ varies smoothly with t . Moreover we have that for each $t \in \mathbb{R}$ $\{t\} \times \Sigma$ is the image through the diffeomorphism of a smooth spacelike Cauchy surface of \mathcal{M} .*

We do not include the proof of this theorem here, however we give some references. That the second condition implies the first is proved in [O’N83, Chap. 14, Cor. 39, p.422]. Moreover in [BS05] Bernal and Sanchez showed that the third condition follows from the first one. With these facts the proof is completed since the implication from the third condition to the second one is trivial.

The next proposition shows that causal convexity entails global hyperbolicity for open subsets of an arbitrary globally hyperbolic spacetime.

Proposition 1.2.16. *Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be a globally hyperbolic spacetime and let Ω be a subset of M . Then if Ω is \mathcal{M} -causally convex, it is also \mathcal{M} -globally hyperbolic.*

Proof. We suppose that Ω is \mathcal{M} -causally convex and we try to show that Ω is also \mathcal{M} -globally hyperbolic. Since \mathcal{M} is a globally hyperbolic spacetime, the g -causality condition is fulfilled by the entire set underlying \mathcal{M} , hence it is fulfilled also by Ω . Then we must only check the other condition for global hyperbolicity. To this end we fix $p, q \in \Omega$. Since causal convexity implies causal compatibility (cfr. Remark 1.2.10), we have that $J_{\pm}^{\mathcal{M}, \Omega}(r) = J_{\pm}^{\mathcal{M}}(r) \cap \Omega$ for each $r \in \Omega$. It follows that

$$J_{+}^{\mathcal{M}, \Omega}(p) \cap J_{-}^{\mathcal{M}, \Omega}(q) = J_{+}^{\mathcal{M}}(p) \cap J_{-}^{\mathcal{M}}(q) \cap \Omega.$$

Since \mathcal{M} is globally hyperbolic, we deduce that $J_{+}^{\mathcal{M}}(p) \cap J_{-}^{\mathcal{M}}(q)$ is compact with respect to the topology of M . If we can show that it is also contained in Ω , then it is compact also with respect to the topology induced on Ω by the topology of M and we also have

$$J_{+}^{\mathcal{M}}(p) \cap J_{-}^{\mathcal{M}}(q) \cap \Omega = J_{+}^{\mathcal{M}}(p) \cap J_{-}^{\mathcal{M}}(q).$$

This would complete the proof. Consider then an arbitrary point r in $J_{+}^{\mathcal{M}}(p) \cap J_{-}^{\mathcal{M}}(q)$. Recalling the definitions of causal future and causal past, we find a \mathfrak{t} -future directed g -causal curve γ_1 in M from p to r and a \mathfrak{t} -past directed g -causal curve γ_2 in M from q to r . Reversing γ_2 and pasting the result with γ_1 , we obtain a \mathfrak{t} -future directed g -causal curve γ in M from p to q . Since p and q are points of Ω and Ω is causally convex, we deduce that γ is entirely contained in Ω . Since r is in the image

of γ , it turns out that $r \in \Omega$ and so the inclusion $J_+^{\mathcal{M}}(p) \cap J_-^{\mathcal{M}}(q) \subseteq \Omega$ actually holds. \square

Remark 1.2.17. Once that a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ is provided, we can build a wide class of \mathcal{M} -causally convex connected open subsets of M that include a Cauchy surface of \mathcal{M} . Applying Theorem 1.2.15 to \mathcal{M} , we obtain a diffeomorphism f that factorizes M into $\mathbb{R} \times \Sigma$ such that $f^{-1}(\{t\} \times \mathbb{R})$ is a smooth spacelike Cauchy surface of \mathcal{M} for each $t \in \mathbb{R}$. Then we can consider $(-\varepsilon, \varepsilon) \times \Sigma$ for an arbitrary $\varepsilon > 0$ and define $\Omega_\varepsilon = f^{-1}((-\varepsilon, \varepsilon) \times \Sigma)$. We immediately deduce that Ω_ε is a connected open subset of M that includes $f^{-1}(\{t\} \times \Sigma)$ for each $t \in (-\varepsilon, \varepsilon)$, which are all smooth spacelike Cauchy surfaces for \mathcal{M} . It remains only to check that Ω_ε is \mathcal{M} -causally convex. Consider a \mathfrak{t} -future directed g -causal curve $\gamma : [a, b] \rightarrow M$ which starts and ends in Ω_ε . Using the factorization of \mathcal{M} in $\mathbb{R} \times \Sigma$ and noting that the projection $\pi_1 : \mathbb{R} \times \Sigma \rightarrow \mathbb{R}$ on the first argument of the Cartesian product $\mathbb{R} \times \Sigma$ is continuous, we deduce that $\pi_1 \circ f \circ \gamma : [a, b] \rightarrow \mathbb{R}$ is continuous. If, by contradiction, along γ there is a point r that is outside Ω_ε , then we find $c \in (a, b)$ such that one of the following inequalities holds:

$$\begin{aligned} (\pi_1 \circ f \circ \gamma)(c) &> \varepsilon > (\pi_1 \circ f \circ \gamma)(b) > (\pi_1 \circ f \circ \gamma)(a); \\ (\pi_1 \circ f \circ \gamma)(c) &< -\varepsilon < (\pi_1 \circ f \circ \gamma)(a) < (\pi_1 \circ f \circ \gamma)(b). \end{aligned}$$

Consider for example the first case (the other one is similar). As a consequence of the intermediate value theorem, we find $d \in [a, c]$ such that

$$(\pi_1 \circ f \circ \gamma)(d) = (\pi_1 \circ f \circ \gamma)(b),$$

which is to say that γ meets twice the smooth spacelike Cauchy surface for \mathcal{M} of the form $\Sigma' = f^{-1}(\{(\pi_1 \circ f \circ \gamma)(b)\} \times \Sigma)$. Exploiting [O'N83, Chap. 14, Lem. 42, p. 425], we find that Σ' is acausal because it is a spacelike Cauchy surface. Then we have found a contradiction, hence γ is contained in Ω_ε and so Ω_ε is actually \mathcal{M} -causally convex.

Indeed there are more powerful constructions that allow us to obtain subsets with good topological and causal properties starting from a globally hyperbolic spacetime. The next proposition is devoted to the recollection of some results that go in this direction.

Proposition 1.2.18. *Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be a globally hyperbolic spacetime.*

- *If Σ is a Cauchy surface for \mathcal{M} and K is a compact subset of M , then both $\Sigma \cap J_\pm^{\mathcal{M}}(K)$ and $J_\pm^{\mathcal{M}}(\Sigma) \cap J_\mp^{\mathcal{M}}(K)$ are compact subsets of M .*
- *If K and K' are compact subsets of M , then $J_\pm^{\mathcal{M}}(K) \cap J_\mp^{\mathcal{M}}(K')$ is a compact subset of M too.*

- If A and B are two non empty subsets of M , then $\Omega = I_+^{\mathcal{M}}(A) \cap I_-^{\mathcal{M}}(B)$ is a \mathcal{M} -causally convex open subset of M . Furthermore if A and B are relatively compact in M , Ω is relatively compact in M too.
- If K is a compact subset of M , then there exists a \mathcal{M} -causally convex relatively compact connected open subset Ω of M including K .

Proof. The proof of the first three points can be found in [BGP07, Cor. A.5.4, p. 175], [BGP07, Lem. A.5.7, p. 176] and [BGP07, Lem. A.5.12, p. 178]. However for the third point the thesis of Bär, Ginoux and Pfäffle is that Ω is \mathcal{M} -globally hyperbolic and \mathcal{M} -causally compatible in place of \mathcal{M} -causally convex. Anyway we can directly check that Ω is \mathcal{M} -causally convex in the following manner. Suppose that γ is a \mathfrak{t} -future directed g -causal curve in M starting from $p \in \Omega$ and ending in $q \in \Omega$. Then each point r along γ is contained in $J_+^{\mathcal{M}}(p) \cap J_-^{\mathcal{M}}(q)$. Since $p \in I_+^{\mathcal{M}}(A)$, we find $p' \in A$ such that $p \in I_+^{\mathcal{M}}(p')$ and, since $q \in I_-^{\mathcal{M}}(B)$, we find $q' \in B$ such that $q \in I_-^{\mathcal{M}}(q')$. From [O'N83, Chap. 14, Cor. 1, p. 402] we deduce that the following implication holds: if $t \in I_{\pm}^{\mathcal{M}}(s)$ and $u \in J_{\pm}^{\mathcal{M}}(t)$, then $u \in I_{\pm}^{\mathcal{M}}(s)$. Hence $J_+^{\mathcal{M}}(p) \subseteq I_+^{\mathcal{M}}(p')$ and $J_-^{\mathcal{M}}(q) \subseteq I_-^{\mathcal{M}}(q')$ so that

$$J_+^{\mathcal{M}}(p) \cap J_-^{\mathcal{M}}(q) \subseteq I_+^{\mathcal{M}}(p') \cap I_-^{\mathcal{M}}(q') \subseteq I_+^{\mathcal{M}}(A) \cap I_-^{\mathcal{M}}(B) = \Omega.$$

This inclusion implies that γ is completely included in Ω .

The proof of the fourth point is obtained modifying in a proper way [BGP07, Lem. A.5.13, p. 178]. As a first step we apply Theorem 1.2.15 to \mathcal{M} and we find a diffeomorphism f that factorizes M in $\mathbb{R} \times \Sigma$ so that $\Sigma_t = f^{-1}(\{t\} \times \Sigma)$ is a smooth spacelike Cauchy surface for \mathcal{M} for each $t \in \mathbb{R}$. The projection $\pi_1 : \mathbb{R} \times \Sigma \rightarrow \mathbb{R}$ on the first factor of the Cartesian product is a continuous map so that $\pi_1 \circ f : M \rightarrow \mathbb{R}$ is continuous. Then the image of the compact subset K of M through $\pi_1 \circ f$ is compact in \mathbb{R} , so that we easily find t_- and t_+ in \mathbb{R} such that $t_- < t < t_+$ for each $t \in (\pi_1 \circ f)(K)$. Now we take $C = J_+^{\mathcal{M}}(K) \cap \Sigma_{t_+}$ and, applying the first point, we conclude that it is a compact subset of M . Hence we can easily find a relatively compact connected open subset A of M including C . Since Σ_{t_+} is a smooth spacelike Cauchy surface for \mathcal{M} , it is easy to check that $K \subseteq J_-^{\mathcal{M}}(C)$. But C is closed since the topology of M is Hausdorff, while A is open by construction and $C \subseteq A$. Then it follows that $J_-^{\mathcal{M}}(C) \subseteq I_-^{\mathcal{M}}(A)$: A point p in $J_-^{\mathcal{M}}(C)$ is connected to a point q of C via a \mathfrak{t} -past directed g -causal curve γ in M starting at q and ending at p ; we can find a neighborhood O of p included in A so that we can deform γ in a way that it becomes timelike in O ; hence we obtain a new \mathfrak{t} -past directed g -causal curve γ' that starts in a point r of $O \subseteq A$ and ends in p and we notice that it cannot be a null curve, i.e. causal, but nowhere timelike, so that we can make a fixed endpoint deformation of γ' (cfr. [O'N83, Chap. 10, Prop. 46, p. 294]) to obtain a \mathfrak{t} -past directed g -timelike curve γ'' that starts in $r \in A$ and ends in p . Returning

to our main proof, we conclude that $K \subseteq I_-^{\mathcal{M}}(A)$. We immediately deduce also that $I_-^{\mathcal{M}}(K) \subseteq I_-^{\mathcal{M}}(A)$. Take now $D = J_-^{\mathcal{M}}(\overline{A}) \cap \Sigma_{t_-}$ and applying again the first point, we find that it is a compact subset of M . Hence we can easily find a relatively compact open subset B of M including D . Keeping in mind that

$$D = J_-^{\mathcal{M}}(\overline{A}) \cap \Sigma_{t_-} \supseteq I_-^{\mathcal{M}}(A) \cap \Sigma_{t_-} \supseteq I_-^{\mathcal{M}}(K) \cap \Sigma_{t_-}$$

and that Σ_{t_-} is a Cauchy surface for \mathcal{M} , we easily check that $\overline{A} \subseteq J_+^{\mathcal{M}}(D)$ and that $K \subseteq I_+^{\mathcal{M}}(D) \subseteq I_+^{\mathcal{M}}(B)$. With a procedure similar to that applied above, we conclude that $\overline{A} \subseteq I_+^{\mathcal{M}}(B)$, hence in particular $A \subseteq I_+^{\mathcal{M}}(B)$. We can apply the second point to the relatively compact open subsets A and B of M and conclude that $\Omega = I_-^{\mathcal{M}}(A) \cap I_+^{\mathcal{M}}(B)$ is a \mathcal{M} -causally convex relatively compact open subset of M . Since by the way we noticed that K is included in both $I_-^{\mathcal{M}}(A)$ and $I_+^{\mathcal{M}}(B)$, we deduce that $K \subseteq \Omega$. The proof is completed if we can show that Ω is connected. To this end take two arbitrary points p and q in Ω . Then we can find a \mathfrak{t} -future directed g -causal curve γ_1 in M that goes from p to some point r in A and a \mathfrak{t} -past directed g -causal curve γ_3 in M that goes from some point s in A to q . Since A is open, $A \subseteq I_-^{\mathcal{M}}(A)$. This fact, together with the inclusion $A \subseteq I_-^{\mathcal{M}}(B)$ shown above, implies that $A \subseteq \Omega$. Hence both γ_1 and γ_3 start and end in Ω . By construction Ω is causally convex and so γ_1 and γ_3 are completely included in Ω . In our construction we choose A to be connected, hence we can find a curve γ_2 from r to s which is included in A (and therefore in Ω too). Pasting γ_1 , γ_2 and γ_3 , we obtain a curve that goes from p to q and the proof of the fourth point is complete. \square

1.3 Wave equations

In this section we face the problem of the existence and uniqueness of global solutions to a given wave equation on a globally hyperbolic spacetime $\mathcal{M} = (M, g)$ with compactly supported smooth initial data on a Cauchy surface Σ of \mathcal{M} . The discussion here involves smooth sections in an arbitrary \mathbb{R} -vector bundle E over M . The results that we recall without proof can be found in [BGP07, Chap. 3]. For a complete discussion on the existence and uniqueness of (local) solutions to wave equations on time oriented Lorentzian manifolds the reader is referred to [BGP07].

1.3.1 Linear differential operators

Since we are going to speak of wave equations in vector bundles over manifolds, we must previously introduce some notions about linear differential operator that will allow us to recognize which differential equations are wave equations in which are not.

Definition 1.3.1. Let M be a d -dimensional manifold and let E and F be two vector bundles over M respectively of rank n and m . A *linear differential operator* L of order at most k from E to F is a \mathbb{R} -linear map

$$L : C^\infty(M, E) \rightarrow C^\infty(M, F)$$

that can be locally written in the following way: For each $p \in M$ there exists an open coordinate neighborhood (U, Ω, ϕ) of p in M on which both E and F are locally trivialized by the maps $\Phi : \pi_E^{-1}(U) \rightarrow U \times \mathbb{R}^n$ and $\Psi : \pi_F^{-1}(U) \rightarrow U \times \mathbb{R}^m$ and there exists a family of local sections $\{A_\alpha \in C^\infty(\Omega, \Omega \times \text{Hom}(\mathbb{R}^n, \mathbb{R}^m))\}_{|\alpha| \leq k}$ such that on Ω we can write

$$\Psi \circ Lu|_U \circ \phi^{-1} = \sum_{|\alpha| \leq k} A_\alpha D_\alpha (\Phi \circ u|_U \circ \phi^{-1}) \quad (1.3.1)$$

for each section $u \in C^\infty(M, E)$, where $\alpha \in \mathbb{N}^d$ is a multi-index, $|\alpha| = \sum_{i=1}^d \alpha_i$, x^1, \dots, x^d are the local coordinates on U and

$$D_\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x^1 \dots \partial^{\alpha_d} x^d}.$$

A *linear differential operator* L of order k is a linear differential operator of order at most k , but not of order at most $k - 1$.

At this point our aim is to identify a specific class of linear differential operators of order 2, but to do this we need to introduce another tool.

Definition 1.3.2. Let M be a d -dimensional manifold and let E and F be two vector bundles over M respectively of rank n and m . Consider a linear differential operator L of order k from E to F . We say that the *principal symbol* σ_L of the linear differential operator L is the map

$$\sigma_L \in T^*M \rightarrow \text{Hom}(E, F)$$

locally defined in a way that is based on Definition 1.3.1: For each $p \in M$ there exists a coordinate neighborhood (U, Ω, ϕ) of p in M on which both E and F are locally trivialized by the maps Φ and Ψ and there exists a family of local smooth sections $\{A_\alpha\}_{|\alpha| \leq k}$ such that on Ω eq. (1.3.1) holds for each section $u \in C^\infty(M, E)$; hence for each $q \in U$, each $\omega \in T_q^*M$ and each $\mu \in E_q$ we set

$$\Psi_q((\sigma_L(\omega))\mu) = \sum_{|\alpha|=k} \omega_1^{\alpha_1} \dots \omega_d^{\alpha_d} (A_\alpha(\phi^{-1}(q))) (\Phi_q \mu),$$

where $\{\omega_1, \dots, \omega_d\}$ are the components of $\omega = \omega_i dx^i$ in the basis $\{dx^1, \dots, dx^d\}$ of $T_p^*M = T_p^*U$ obtained via pull back through ϕ from the orthonormal basis

$\{e_1, \dots, e_d\}$ of $\mathbb{R}^d = T_{\phi(p)}^* \Omega$.

Example 1.3.3. The formulation of the last definitions may appear very abstract (at least this was the impression of the author when he saw them for the first time), but they are much more concrete and close to the usual idea of partial derivative than it seems. However to realize this fact we must restrict ourselves to a more customary situation. Consider for example $M = \mathbb{R}^d$ and $E = M \times \mathbb{R}^n$ and $F = M \times \mathbb{R}^m$. In this case there are a global coordinate neighborhood for M and global trivializations for E and F , while TM reduces to $M \times \mathbb{R}^d$ so that we can identify it with T^*M . Moreover a section u in E is nothing but an \mathbb{R}^n -valued smooth function defined on $M = \mathbb{R}^d$. In this situation one recognizes that partial derivatives of order at most k and their linear combinations with $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ -valued smooth functions defined on M as coefficients are undoubtedly linear differential operators from E to F of order at most k . If there is a partial derivative of order k with non null coefficient, the operator is exactly of order k . The local sections $\{A_\alpha\}$ in this case are actually global and coincide with the $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ -valued smooth functions defined on M that we used as coefficients. The principal symbol is simply a function from $T^*M = M \times \mathbb{R}^d$ to $\text{Hom}(E, F) = M \times \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ that maps each $(x, \omega) = ((x^1, \dots, x^d), (\omega_1, \dots, \omega_d)) \in T^*M$ to a linear combination of the coefficients A_α corresponding to the derivatives of highest order weighted with products of the components of ω with powers that equal the order of the partial derivatives along each direction. For example,

$$\begin{aligned} & \begin{pmatrix} e^{x^1 x^4} & 0 & \cos x^4 \\ 4 & \tanh x^3 & 7 \end{pmatrix} \frac{\partial^4}{\partial x^1 \partial (x^3)^3} + \begin{pmatrix} (x^2)^2 & 1 & 0 \\ 3 & x^3 & -1 \end{pmatrix} \frac{\partial^4}{\partial (x^2)^2 \partial (x^4)^2} \\ & + \begin{pmatrix} 5 & \sinh(x^2 x^3) & 0 \\ 0 & x^1 + x^3 & \frac{x^4}{2} \end{pmatrix} \frac{\partial^3}{\partial x^1 \partial x^2 \partial x^3} + \begin{pmatrix} x^2 & 1 & \sin x^1 \\ 3 & 0 & x^4 - x^2 \end{pmatrix} \frac{\partial^2}{\partial (x^4)^2} \end{aligned}$$

is a linear differential operator from $E = M \times \mathbb{R}^2$ to $F = M \times \mathbb{R}^3$ of order 4, where $M = \mathbb{R}^4$, whose principal symbol is the map

$$\sigma_L : T^*M = M \times \mathbb{R}^4 \rightarrow \text{Hom}(E, F) = M \times \text{Hom}(\mathbb{R}^2, \mathbb{R}^3)$$

defined by

$$\sigma_L(x, \omega) = \omega_1 \omega_3^3 \begin{pmatrix} e^{x^1 x^4} & 0 & \cos x^4 \\ 4 & \tanh x^3 & 7 \end{pmatrix} + \omega_2^2 \omega_4^2 \begin{pmatrix} (x^2)^2 & 1 & 0 \\ 3 & x^3 & -1 \end{pmatrix}.$$

We conclude this subsection with the notion of formal selfadjointness.

Definition 1.3.4. Let (M, \mathfrak{o}) be a d -dimensional oriented manifold endowed with a metric g and let E be a vector bundle over M of rank n endowed with an inner product that we denote with $\overset{E}{\cdot, \cdot}$. A linear operator $L : C^\infty(M, E) \rightarrow C^\infty(M, E)$ is

said to be *formally selfadjoint* if for each $u, v \in \mathcal{D}(M, E)$ we have

$$\int_M (Lu)^E \cdot v d\mu_g = \int_M u^E \cdot (Lv) d\mu_g,$$

where $d\mu_g$ is volume form over (M, \mathfrak{o}) induced by g . If the equation above holds with a minus sign at the RHS then L is *formally antiselfadjoint*.

Indeed we will apply the last definition to linear differential operators, but more in general it can be applied to operators acting linearly on smooth sections in a vector bundle.

1.3.2 Normally hyperbolic equations and Cauchy problems

We are ready to pick out a particular class of linear differential operators of order 2 which are at the core of the theory of wave equations on globally hyperbolic spacetimes. Probably the reader has some notion of what it is generally meant as a wave equation. However, in the present context, for wave equation we intend a class of linear differential equations of second order that may be a little bit larger than what it is usually intended. In order to avoid misunderstanding, we take the chance to define our notion of wave equation (to be more precise, of normally hyperbolic equation).

Definition 1.3.5. Let (M, g) be a Lorentzian manifold and let E be a vector bundle over M of rank n . A *normally hyperbolic operator* P on E over (M, g) is a linear differential operator of order 2 from E to E whose principle symbol σ_P is of *metric type*, i.e. for each $p \in M$ and each $\omega \in T_p^*M$

$$\sigma_P(p, \omega) = -g_p(\omega^\sharp, \omega^\sharp) \text{id}_{E_p},$$

where $^\sharp : T^*M \rightarrow TM$ is the raising isomorphism induced by the metric g (see Definition 1.1.27).

A *normally hyperbolic equation* (or *wave equation*) on a vector bundle E over a Lorentzian manifold (M, g) is a linear differential equation of the form

$$Pu = v,$$

where P is a normally hyperbolic operator on E over (M, g) and u is a smooth section in E over M to be determined, while $v \in C^\infty(M, E)$ is given.

Example 1.3.6. Consider the Minkowski spacetime, i.e. the manifold $M = \mathbb{R}^4$ endowed with a metric g that is everywhere represented by the matrix $(g_{ij}) = \text{diag}(-1, +1, +1, +1)$, and the vector bundle $E = M \times \mathbb{R}^4$. Recalling our Example

1.3.3, we see that the Klein-Gordon operator on Minkowski spacetime

$$-g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + m^2 \text{id}_{C^\infty(M, E)} : C^\infty(M, E) \rightarrow C^\infty(M, E),$$

where (g^{ij}) is the inverse of the matrix (g_{ij}) and $m \geq 0$ is a parameter (the mass of the Klein-Gordon field), is a linear differential operator from E to E of order 2. Its principal symbol is provided by the function that maps each $(x, \omega) \in T^*M$ to

$$\begin{aligned} -\omega_i \omega_j g^{ij} \text{id}_E &= -(\omega^\sharp)^k g_{ki} (\omega^\sharp)^h g_{hj} g^{ij} \text{id}_E = -(\omega^\sharp)^k (\omega^\sharp)^h g_{kh} \text{id}_E \\ &= -g_x(\omega^\sharp, \omega^\sharp) \text{id}_E, \end{aligned}$$

hence we recognize that the Klein-Gordon operator in Minkowski spacetime is a normally hyperbolic operator.

Maybe the most common prototype of wave equation is the d'Alembert equation. The d'Alembert operator

$$\begin{aligned} \square^\nabla : C^\infty(M, E) &\rightarrow C^\infty(M, T^*M \otimes T^*M \otimes E) \\ u &\mapsto (-(\text{tr}_{T^*M} \otimes \text{id}_E) \circ \nabla \circ \nabla) u \end{aligned}$$

induced by a connection ∇ on a vector bundle E over a Lorentzian manifold (M, g) is indeed a normally hyperbolic operator on E over (M, g) (for a proof of this fact refer to [BGP07, Ex. 1.5.2, p. 34]) and hence the d'Alembert equation is a normally hyperbolic equation on E over (M, g) .

However there exist many other normally hyperbolic equations. For example notice that each equation involving the d'Alembert operator defined above together with other linear differential terms of order at most 1 is still a normally hyperbolic equation.

It can be even shown that each normally hyperbolic operator P on a vector bundle E over a Lorentzian manifold (M, g) can be written as the sum of the d'Alembert operator \square^∇ associated to some connection ∇ on E with a section B in $\text{End}(E, E)$ (cfr. [BGP07, Lem. 1.5.5, p. 35]). In this case the connection ∇ is called *P-compatible*.

These observations are made to underline that the typical wave equations are indeed included in our class of normally hyperbolic equation, but there are also other (although quite similar) partial differential equations that fall in our class.

We have defined all the ingredients needed to state a theorem about the existence and uniqueness of solutions for a non homogeneous normally hyperbolic equation.

Theorem 1.3.7. *Consider a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ and a vector bundle E over M . Let Σ be a spacelike smooth Cauchy surface of \mathcal{M} , let $\mathfrak{n} \in C^\infty(\Sigma, TM)$ be a unit \mathfrak{t} -future directed g -timelike vector field over Σ normal to*

Σ and let P be a normally hyperbolic operator on (E, \mathcal{M}) . Denote the P -compatible connection with ∇ . Then for each $f \in \mathcal{D}(M, E)$ and each $u_0, u_1 \in \mathcal{D}(\Sigma, \pi_E^{-1}(\Sigma))$ there exists a unique solution $u \in C^\infty(M, E)$ to the Cauchy problem

$$\begin{cases} Pu &= f, \\ u|_\Sigma &= u_0, \\ \nabla_n u|_\Sigma &= u_1. \end{cases}$$

Moreover we have $\text{supp}(u) \subseteq J(K)$, where $K = \text{supp}(u_0) \cup \text{supp}(u_1) \cup \text{supp}(f)$.

The proof of the last theorem is based on the determination of the so called fundamental solutions. Even if we do not discuss here such proof, it is useful for us to introduce fundamental solutions in view of the construction of Green operators. We face these problems after having introduced the necessary material, specifically distributions on manifolds. To such topic we devote the next subsection.

1.3.3 Distributions on manifolds

To introduce the notion of fundamental solution we cannot restrict to sections over vector bundles. We need to introduce “sections” in a broader sense. Distributions on manifolds are the right tools for our aims. Before we define such objects we need to provide a notion of convergence in $\mathcal{D}(M, E)$. This requires some preparation.

Let M be a manifold with a Riemannian metric $g \in C^\infty(M, T^*M \otimes_s T^*M)$ and let E be a vector bundle over M endowed with a connection

$$\nabla : C^\infty(M, E) \rightarrow C^\infty(M, T^*M \otimes E)$$

and a positive definite inner product $h \in C^\infty(M, E^* \otimes_s E^*)$. Use again ∇ to denote the Levi-Civita connection on TM :

$$\nabla : C^\infty(M, TM) \rightarrow C^\infty(M, T^*M \otimes TM).$$

Notice that ∇ and g induce (via duality and tensor product) a connection

$$\nabla : C^\infty(M, T^{(i,j)}M) \rightarrow C^\infty(M, T^*M \otimes T^{(i,j)}M)$$

and respectively a positive definite inner product $g \in C^\infty(M, T^{(j,i)}M \otimes_s T^{(i,j)}M)$ on each $T^{(i,j)}M$. Putting together the connections and the inner products on $T^{(i,j)}M$ and E we can obtain (via tensor product) a connection

$$\nabla : C^\infty(M, T^{(i,j)}M \otimes E) \rightarrow C^\infty(M, T^*M \otimes T^{(i,j)}M \otimes E)$$

and respectively an inner product

$$k \in C^\infty(M, (T^{(j,i)}M \otimes E^*) \otimes (T^{(j,i)}M \otimes E^*))$$

on each vector bundle $T^{(i,j)}M \otimes E$. For each $p \in M$, k induces a norm $|\cdot|_p$ on the fiber E_p defined by

$$|\mu|_p^2 = \mu \cdot_{k,p} \mu$$

for each $\mu \in T_p^{(i,j)}M \otimes E_p$. Then we can use the collection $\{|\cdot|_p : p \in M\}$ of fiberwise norms and the connection in $T^{(i,j)}M \otimes E$ to define a family of seminorms on the space $C^\infty(M, E)$: For each compact subset K of M we set

$$|u|_K = \sup_{i \in \mathbb{N}} \left(\sup_{p \in K} |\nabla^i u|_p \right),$$

where $\nabla^i u$ means the application of the connection

$$\nabla : C^\infty(M, T^{(0,i-1)}M \otimes E) \rightarrow C^\infty(M, T^{(0,i)}M \otimes E)$$

to the C^∞ -section $\nabla^{i-1}u$.

Definition 1.3.8. Let M be a manifold endowed with a metric g and let E be a vector bundle over M endowed with a connection ∇ and an inner product h . Endow TM with the Levi-Civita connection still denoted by ∇ . Following the construction above we define a notion of convergence in $\mathcal{D}(M, E)$: We say that a sequence $\{u_i\} \subseteq \mathcal{D}(M, E)$ converges to $u \in \mathcal{D}(M, E)$ if there exists a compact subset K of M such that $\text{supp}(u_n)$ and $\text{supp}(u)$ are contained in K for each $i \in \mathbb{N}$ and the sequence $\{|u_i - u|_K\}$ converges to zero.

Notice that, since we always consider compact subsets, it can be proved that different choices of inner products (provided that they are positive definite) and connections yield equivalent seminorms, hence the notion of convergence the we defined on $\mathcal{D}(M, E)$ does not depend on the choices made in the preparatory construction.

Now we can speak of distributions on manifolds.

Definition 1.3.9. Consider a manifold M , a vector bundle E over M and a finite dimensional \mathbb{R} -vector space V . A V -valued distribution in E is a linear map $U : \mathcal{D}(M, E^*) \rightarrow V$ that is continuous with respect to the convergence in $\mathcal{D}(M, E^*)$.

$\mathcal{D}'(M, E, V)$ denotes the vector space of V -valued distributions in E .

In the definition given above the choice of a norm on the vector space V is implied. However $\dim V < \infty$, hence all norms are equivalent and hence the definition does not depend on the choice of the norm on V .

Remark 1.3.10. Consider an oriented manifold (M, \mathfrak{o}) endowed with a metric g , two vector bundles E and F over M and a finite dimensional \mathbb{R} -vector space V . There is

a procedure to extend any linear differential operator $L : C^\infty(M, E) \rightarrow C^\infty(M, F)$ to a *linear differential operator in distributional sense*, that is a linear map from $\mathcal{D}'(M, E, V)$ to $\mathcal{D}'(M, F, V)$ which we still denote with L .

The first thing to be done is to define the *formal adjoint* of L , denoted by L^* . Precisely, there exists a unique linear differential operator $L^* : C^\infty(M, F^*) \rightarrow C^\infty(M, E^*)$ such that

$$\int_M (L^*u)(v) d\mu_g = \int_M u(Lv) d\mu_g$$

for each $u \in \mathcal{D}(M, F^*)$ and each $v \in \mathcal{D}(M, E)$, where the dual pairing between the proper vector bundles is taken into account and $d\mu_g$ is the volume form induced by g on (M, \mathfrak{o}) . Notice that the canonical identification $(E^*)^* = E$ implies $(L^*)^* = L$, where for $(L^*)^*$ we mean the formal adjoint of L^* defined repeating the procedure just shown.

At this point we are ready to extend L^* to a linear operator from $\mathcal{D}'(M, E, V)$ to $\mathcal{D}'(M, F, V)$, that we denote again with L . This is the linear differential operator in distributional sense that extends the “original” L . Such extension is obtained imposing

$$(LU)[v] = U[L^*v]$$

for each $U \in \mathcal{D}'(M, E, V)$ and each $v \in \mathcal{D}(M, F^*)$.

Note that, in the case $V = \mathbb{R}$, the “new” L acts exactly as the “original” L on sections of $C^\infty(M, E)$ (to be precise, we should say that, for each $u \in C^\infty(M, E)$, there exists a unique section $v \in C^\infty(M, F)$ that generates the image through the “new” L of the distribution generated by u and that such v coincides with the image through the “original” L of u). This fact is a consequence of the identity $(L^*)^* = L$ shown above.

Before proceeding with the next subsection, we want to make some remarks about formally selfadjoint linear differential operators and their extensions in distributional sense.

Remark 1.3.11. Assume that E is a vector bundle over an oriented manifold (M, \mathfrak{o}) endowed with a metric g and consider an inner product on E . Suppose that $L : C^\infty(M, E) \rightarrow C^\infty(M, E)$ is a formally selfadjoint linear differential operator. Considering the musical isomorphisms defined using the inner product on E (cfr. Definition 1.1.27), we realize that the condition of formal selfadjointness (cfr. Definition 1.3.4) can be rewritten in the following form:

$$\int_M (Lu)^\flat(v) d\mu_g = \int_M u^\flat(Lv) d\mu_g \quad \forall u, v \in \mathcal{D}(M, E),$$

where $d\mu_g$ is the volume form induced by g on (M, \mathfrak{o}) . In the present situation the

formal adjoint of L is given by $\flat \circ L \circ \sharp$. We can check this fact verifying that, because of the formal selfadjointness of L , $\flat \circ L \circ \sharp$ satisfies the formula, given in our last remark, that defines uniquely the formal adjoint of a linear differential operator: for each $u \in \mathcal{D}(M, E^*)$ and each $v \in \mathcal{D}(M, E)$ we have

$$\int_M ((\flat \circ L \circ \sharp) u)(v) d\mu = \int_M (L(u^\sharp))^\flat(v) d\mu = \int_M (u^\sharp)^\flat(Lv) d\mu = \int_M u(Lv) d\mu.$$

At this point we have $L^* = \flat \circ L \circ \sharp$. Now identify E and E^* through the musical isomorphisms and we deduce $L^* = L$. Then, after the identification of E^* with E as done before, formal selfadjointness of L means that the formal adjoint of L coincides with L . This fact trivially leads also to the coincidence of the extensions of L and L^* as linear differential operators in distributional sense.

1.3.4 Fundamental solutions and Green operators

Once that we are given a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, a vector bundle E over M , a normally hyperbolic operator $P : C^\infty(M, E) \rightarrow C^\infty(M, E)$ on (E, \mathcal{M}) and a vector space V , applying Remark 1.3.10, the distributional extension $P : \mathcal{D}'(M, E, V) \rightarrow \mathcal{D}'(M, E, V)$ of the “original” P . For our current scope, that is the determination of global fundamental solutions for each point of M , we need to consider a different vector space each time and hence we have to define a “new” P for each $p \in M$. The reason that induces us to do this will become clear in the next definition.

Definition 1.3.12. Consider a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, a vector bundle E over M and a linear differential operator $P : C^\infty(M, E) \rightarrow C^\infty(M, E)$ on E over \mathcal{M} . Then for each $p \in M$ we consider the linear differential operator in distributional sense $P : \mathcal{D}'(M, E, E_p^*) \rightarrow \mathcal{D}'(M, E, E_p^*)$ (obtained from the given P exploiting Remark 1.3.10) and we call *fundamental solution for P at the point p* each of the distributions of $\mathcal{D}'(M, E, E_p^*)$ that solve the equation $PU = \delta_p$ in distributional sense, where $\delta_p : \mathcal{D}(M, E^*) \rightarrow E_p^*$ is the E_p^* -valued delta distribution at p on E over M defined by $\delta_p[w] = w(p)$ for each $w \in \mathcal{D}(M, E^*)$.

Now the reason for which we consider a different “new” P for each $p \in M$ becomes clear: once that $p \in M$ is fixed, we have to consider E_p^* as target vector space for the space of distributions in which we search fundamental solutions in order to get compatibility between the RHS and the LHS of the distributional equation $PU = \delta_p$.

We have defined fundamental solutions at a given point. Now we have the problem of their existence and, in case, their uniqueness. The next theorem provides us a tool that ensures uniqueness under certain hypotheses.

Lemma 1.3.13. Consider a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, a vector bundle E over M , a vector space V and a normally hyperbolic operator P on E over

\mathcal{M} . Then each solution $u \in \mathcal{D}'(M, E, V)$ of the equation $Pu = 0$ (in distributional sense) with past compact or future compact support must identically vanish.

We stress that this lemma guarantees uniqueness only for fundamental solutions with past compact or future compact support. Nothing is implied for fundamental solutions with different supports.

The hypothesis of global hyperbolicity in this lemma can be weakened without modifying the thesis. We keep such hypothesis also here since anyway throughout our discussion it will always be assumed being indispensable for many essential results, e.g. the next theorem, in which a relaxation of the hypothesis of global hyperbolicity leads to the loss of the result of existence for global fundamental solutions with past compact or future compact support.

In the statement of the next theorem, besides existence for global fundamental solutions with past compact or future compact support, we have included uniqueness too, which is a direct consequence of Lemma 1.3.13.

Theorem 1.3.14. *Consider a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, a vector bundle E over M and a normally hyperbolic operator P on E over \mathcal{M} . Then for each $p \in M$ there exists exactly one fundamental solution for P at p with past compact support (we denote it with $U^a(p)$) and exactly one fundamental solution for P at p with future compact support (we denote it with $U^r(p)$). Such fundamental solutions satisfy the following properties:*

- $\text{supp}(U^a(p)) \subseteq J_+(p)$ and $\text{supp}(U^r(p)) \subseteq J_-(p)$;
- for each $u \in \mathcal{D}(M, E^*)$ the maps $p \mapsto U^{a/r}(p)[u]$, denoted by $U^{a/r}(\cdot)[u]$, are (smooth) sections in E^* over M and satisfy the differential equation

$$P^*(U^{a/r}(\cdot)[u]) = u.$$

Beyond Lemma 1.3.13, the proof of this last result requires Theorem 1.3.7 and another theorem (not included here) that guarantees the linearity and continuity of the map that associates to each proper initial data the corresponding solution of the Cauchy problem presented in Theorem 1.3.7. Such theorem can be found in [BGP07, Thm. 3.2.12, p. 86]. Both Theorem 1.3.7 and the omitted theorem are applied to P^* in place of P . The hypotheses of these theorems require that P^* is normally hyperbolic. Indeed this follows from the hypothesis of normal hyperbolicity of P .

Now we can use fundamental solutions and their properties to define a pair of operators that will allow us to obtain the full set of solutions of the homogeneous Cauchy problems with compactly supported initial data starting from the space of compactly supported sections. We begin with a definition.

Definition 1.3.15. Consider a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, a vector bundles E over M and a linear differential operator $P : C^\infty(M, E) \rightarrow C^\infty(M, E)$. We call *advanced Green operator for P* each map

$$e^a : \mathcal{D}(M, E) \rightarrow C^\infty(M, E)$$

that is linear and fulfils the following requirements for each $f \in \mathcal{D}(M, E)$:

- $Pe^a f = f$;
- $e^a P f = f$;
- $\text{supp}(e^a f) \subseteq J_+(\text{supp}(f))$.

Similarly, we call *retarded Green operator for P* each map

$$e^r : \mathcal{D}(M, E) \rightarrow C^\infty(M, E)$$

that is linear and fulfils the same requirements with J_+ replaced by J_- .

Theorem 1.3.14 implies existence and uniqueness of both an advanced Green operator and a retarded Green operator for a normally hyperbolic operator (we call them respectively the advanced Green operator and the retarded Green operator since they are unique). We present such result in the following corollary.

Corollary 1.3.16. *Consider a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, a vector bundle E over M and a normally hyperbolic operator P on E over \mathcal{M} . Then two families $\{U^a(x)\}$ and $\{U^r(x)\}$ of fundamental solutions for P^* with past compact and, respectively, future compact support define advanced and retarded Green operators e^a and e^r for P in the following way: $e^a f = U^r(\cdot)[f]$ and $e^r f = U^a(\cdot)[f]$ for each $f \in \mathcal{D}(M, E)$.*

Also the converse is true, i.e. advanced and retarded Green operators e^a and e^r for P define two families $\{U^a(x)\}$ and $\{U^r(x)\}$ of fundamental solutions for P^ with past compact and, respectively, future compact support through the formulas given above applied in reverse sense.*

In particular it follows that uniqueness for fundamental solutions with past/future compact support implies uniqueness for Green operators.

The existence of two families of fundamental solutions with the proper support properties is assured by Theorem 1.3.14 applied to P^* , which is normally hyperbolic because we supposed that P is normally hyperbolic. As for uniqueness of Green operators, if we assume that there exist two pairs of Green operators, we can obtain two pairs of families of fundamental solutions with the right support properties. Then Lemma 1.3.13 implies the coincidence of the new families with the original

ones and this fact in turn implies the coincidence of the Green operators used to build such families of fundamental solutions.

We devote the last part of this section to the presentation of some properties related to the Green operators. The first one is an extension of the second property in Definition 1.3.15. Its validity is essentially based on Lemma 1.3.13.

Lemma 1.3.17. *Consider a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, a vector bundle E over M , a normally hyperbolic operator P on E over \mathcal{M} and denote with e^a and e^r its advanced and retarded Green operators. Then for all $u \in C^\infty(M, E)$ such that $Pu \in \mathcal{D}(M, E)$ it holds that:*

- if u has past compact support, $e^a Pu = u$;
- if u has future compact support, $e^r Pu = u$.

Proof. Fix $u \in C^\infty(M, E)$ with past compact support such that $Pu \in \mathcal{D}(M, E)$. Then Pu is in the domain of e^a and hence we can consider $e^a Pu$. From the first property in Definition 1.3.15 we deduce $Pe^a Pu = Pu$ and this identity can be rewritten in the form $P(e^a Pu - u) = 0$. We observe that $\text{supp}(e^a Pu) \subseteq J_+(\text{supp}(Pu))$ and that $J_+(\text{supp}(Pu))$ is past compact (this follows from Proposition 1.2.18). Since u has past compact support by hypothesis, we deduce that $e^a Pu - u$ has past compact support. Consider the distribution $F \in \mathcal{D}'(M, E, \mathbb{R})$ generated by the section $e^a Pu - u$:

$$\begin{aligned} F : \mathcal{D}(M, E^*) &\rightarrow \mathbb{R} \\ f &\mapsto \int_M f(e^a Pu - u) d\mu_g, \end{aligned}$$

where $d\mu_g$ denotes the volume form on \mathcal{M} and the dual pairing between E^* and E is taken into account. We obtain $PF = 0$ in distributional sense: for each $f \in \mathcal{D}(M, E^*)$

$$(PF)[f] = F[P^* f] = \int_M (P^* f)(e^a Pu - u) d\mu_g = \int_M f(P(e^a Pu - u)) d\mu_g = 0.$$

Therefore Lemma 1.3.13 entails that F is the null distribution. Since the only section that generates the null distribution is the null section, we conclude that $e^a Pu - u$ vanishes everywhere, which is to say $e^a Pu = u$. The proof of $e^r Pu = u$ for u with future compact support is similar. \square

Before the definition of Green operators, we have anticipated that they allow us to build the full space of solutions of the homogeneous Cauchy problems for a normally hyperbolic equation with compactly supported initial data. Now we see how this is obtained.

Definition 1.3.18. Consider a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, a vector bundle E over M and a linear differential operator $P : C^\infty(M, E) \rightarrow C^\infty(M, E)$ admitting advanced and retarded Green operators e^a and e^r . We call *causal propagator for P* the operator $e = e^a - e^r$.

The support properties of the advanced and retarded Green operators explain the reason why the operator $e = e^a - e^r$ is called causal propagator for P : one may say that e “propagates” each compactly supported section f to the causal future and past of its support providing a section ef whose support is contained in $J(\text{supp}(f))$.

Corollary 1.3.19. Consider a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, a vector bundle E over M and a normally hyperbolic operator P on E over \mathcal{M} . Let e^a and e^r be the advanced and retarded Green operators for P . Then the space of solutions \mathcal{S} of the homogeneous Cauchy problems associated to P with compactly supported initial data coincides with the image through the causal propagator e of $\mathcal{D}(M, E)$.

Proof. Before starting with the main part of the proof, we notice that we can find a spacelike smooth Cauchy surface Σ of \mathcal{M} since \mathcal{M} is a globally hyperbolic spacetime (see Theorem 1.2.15). We set a unit future directed timelike vector field \mathbf{n} over Σ which is normal to Σ .

We begin from the inclusion $e(\mathcal{D}(M, E)) \subseteq \mathcal{S}$. Fix $f \in \mathcal{D}(M, E)$ and define $u_0 = ef|_\Sigma$ and $u_1 = \nabla_{\mathbf{n}}(ef)|_\Sigma$. $u_0, u_1 \in \mathcal{D}(\Sigma, \pi^{-1}(\Sigma))$: $\text{supp}(ef) \cap \Sigma$ is compact because it is closed and contained in $J(\text{supp}(f)) \cap \Sigma$ which is compact too (cfr. Proposition 1.2.18). The first defining property of Green operators (see Definition 1.3.15) implies trivially that $P(ef) = 0$. Moreover $ef|_\Sigma = u_0$ and $\nabla_{\mathbf{n}}(ef)|_\Sigma = u_1$ by construction, where u_0 and u_1 are proper initial data for a homogeneous Cauchy problem associated to P . Hence $ef \in \mathcal{S}$.

Now we turn our attention to the converse inclusion, i.e. $\mathcal{S} \subseteq e(\mathcal{D}(M, E))$. Fix $u \in \mathcal{S}$. Since u is a solution of a homogeneous Cauchy problem associated to P with compactly supported initial data, we find proper initial data that generate such solution simply imposing $u_0 = u|_\Sigma$ and $u_1 = \nabla_{\mathbf{n}}u|_\Sigma$. As above $u_0, u_1 \in \mathcal{D}(\Sigma, \pi^{-1}(\Sigma))$ because from 1.3.7 it follows that there exists a compact subset K of M such that $\text{supp}(u) \subseteq J(K)$. Therefore we have that u is the unique solution of the following homogeneous Cauchy problem:

$$\begin{cases} Pu &= 0, \\ u|_\Sigma &= u_0, \\ \nabla_{\mathbf{n}}u|_\Sigma &= u_1. \end{cases}$$

It is easy to find a compact subset K' of M that includes the supports of u_0 and u_1 and a relatively compact open subset Ω of M that includes K . We define the open subsets $\Omega_\pm = J_\pm(\Omega)$ and $\Omega_0 = M \setminus J(K')$ of M ($J_\pm(\Omega)$ are open subsets of

M as it is shown in [FV11, Lem. A.8, p. 48], while $J_{\pm}(K)$ are closed subsets of M as it is shown in [BGP07, Lem. A.5.1, p. 173]) and we consider the open covering $\{\Omega_+, \Omega_-, \Omega_0\}$ of M . Associated to such open covering, we can choose a partition of unity $\{\chi_+, \chi_-, \chi_0\}$. We set $v_{\pm} = \chi_{\pm}u \in C^{\infty}(M, E)$ and $v_0 = \chi_0u \in C^{\infty}(M, E)$. From Theorem 1.3.7 we deduce that $\text{supp}(u) \subseteq J(K')$ because K' includes the supports of u_0 and u_1 . Hence $v_0 = 0$ and therefore $Pv_+ = -Pv_-$. In particular this implies that Pv_+ is supported in

$$\text{supp}(\chi_+) \cap \text{supp}(\chi_-) \subseteq J_+(\Omega) \cap J_-(\Omega) \subseteq J_+(\overline{\Omega}) \cap J_-(\overline{\Omega}).$$

Since Ω is relatively compact in M , $J_+(\overline{\Omega}) \cap J_-(\overline{\Omega})$ is compact (cfr. Proposition 1.2.18) and hence $Pv_+ \in \mathcal{D}(M, E)$. Consider now ePv_+ :

$$ePv_+ = e^aPv_+ - e^rPv_+ = e^aPv_+ + e^rPv_- = v_+ + v_- = u,$$

where we applied Lemma 1.3.17 taking into account that v_+ has past compact support and v_- has future compact support as a consequence of their definitions. This completes the proof. \square

The next proposition provides a characterization of the kernel of the causal propagator. The proof is based on the defining properties of Green operators.

Proposition 1.3.20. *Consider a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, a vector bundle E over M and a linear differential operator $P : C^{\infty}(M, E) \rightarrow C^{\infty}(M, E)$ admitting advanced and retarded Green operators e^a and e^r . Then we have that the kernel of the causal propagator e coincides with the image through P of $\mathcal{D}(M, E)$:*

$$\ker e = P(\mathcal{D}(M, E)).$$

Proof. The inclusion $P(\mathcal{D}(M, E)) \subseteq \ker e$ is a trivial consequence of the second defining property of Green operators (cfr. Definition 1.3.15). To prove the converse inclusion, consider $f \in \mathcal{D}(M, E)$ such that $ef = 0$. We have to find $h \in \mathcal{D}(M, E)$ such that $Ph = f$ to prove that f falls in $P(\mathcal{D}(M, E))$. We do this in the following way: First we notice that $ef = 0$ implies $e^af = e^rf$. From this it follows that

$$\text{supp}(e^af) \subseteq J_+(\text{supp}(f)) \cap J_-(\text{supp}(f)).$$

The set on the RHS of the last inclusion is compact owing to Proposition 1.2.18, hence e^af has compact support. Moreover $Pe^af = f$ because of the first property in Definition 1.3.15. Hence we have found a section of the type required: $h = e^af$. \square

The last proposition of this subsection establishes a relationship that holds between the Green operators for a normally hyperbolic operator and the Green operators for its formal adjoint (that is automatically normally hyperbolic).

Proposition 1.3.21. *Consider a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, a vector bundle E over M and a normally hyperbolic operator P on E over \mathcal{M} . Let e^a and e^r be the advanced and retarded Green operators for P and e^{*a} and e^{*r} be the advanced and retarded Green operators for P^* , the formal adjoint of P (cfr. Remark 1.3.10), which is automatically normally hyperbolic. Then we have that $e^{*a/r}$ is formally adjoint to $e^{r/a}$, which is to say*

$$\int_M (e^{*a/r} f)(h) d\mu_g = \int_M f(e^{r/a} h) d\mu_g$$

for each $f \in \mathcal{D}(M, E^*)$ and each $h \in \mathcal{D}(M, E)$, where the dual pairing between E^* and E is taken into account and $d\mu_g$ is the volume form on \mathcal{M} .

Proof. For each $f \in \mathcal{D}(M, E^*)$ and each $h \in \mathcal{D}(M, E)$ we have

$$\begin{aligned} \int_M (e^{*a/r} f)(h) d\mu_g &= \int_M (e^{*a/r} f)(Pe^{r/a} h) d\mu_g = \int_M (P^* e^{*a/r} f)(e^{r/a} h) d\mu_g \\ &= \int_M f(e^{r/a} h) d\mu_g. \end{aligned}$$

In the last calculation we have used the first defining property of Green operators and we have exploited the relation of formal adjointness between P^* and P noting that

$$\text{supp}(e^{*a/r} f) \cap \text{supp}(e^{r/a} h) \subseteq J_{\pm}(\text{supp}(f)) \cap J_{\mp}(\text{supp}(h))$$

is compact due to Proposition 1.2.18. □

1.4 Algebras and states

To define a quantum field theory in a proper way, we need essentially two different types of ingredients. On the one hand there are algebras, whose elements play the role of abstract “quantum observables”. On the other hand there are states, which contain all the information pertaining to the physical system that they are expected to describe. The quantum field theory of a physical system concretely arises only from the interaction of such building blocks. By this we mean that a physical prediction is obtained taking the expectation value of an observables on a given state. This section is devoted to a brief presentation of both these ingredients with particular attention to the algebras that are needed for the quantization of bosonic fields.

1.4.1 C*-algebras, Weyl systems and CCR representations

In this subsection we recollect the essential algebraic equipment that we will use in the next chapters to build quantum field theories. As for Section 1.3, most of the theorems are stated without proofs, however these can be found in [BGP07, Chap. 4, Sects. 1-2].

We begin giving the definition of an algebra. We take the chance to specify some particular types of algebras which are enriched with some additional structures such as *-algebras and C*-algebras

Definition 1.4.1. An *associative \mathbb{C} -algebra* (or simply an *algebra*) \mathcal{A} is a \mathbb{C} -vector space V endowed with a map $V \times V \rightarrow V$, called *multiplication*, that maps $(a, b) \in V \times V$ to an element of V denoted by ab and that fulfils the following properties:

- *\mathbb{C} -bilinearity*: for each $a, b, c \in V$ and each $\eta, \xi \in \mathbb{C}$ it holds that

$$\begin{aligned} (\eta a + \xi b) c &= \eta ac + \xi bc, \\ a (\eta b + \xi c) &= \eta ab + \xi ac; \end{aligned}$$

- *associativity*: $(ab)c = a(bc)$ for each $a, b, c \in V$.

A **-algebra* \mathcal{A} is an algebra endowed with a map $*$: $V \rightarrow V$, called *involution*, that maps $a \in V$ to an element of V denoted by a^* and that fulfils the following properties:

- *involution property*: $a^{**} = a$ for each $a \in V$;
- *\mathbb{C} -antilinearity*: $(\eta a + \xi b)^* = \bar{\eta} a^* + \bar{\xi} b^*$ for each $a, b \in V$ and each $\eta, \xi \in \mathbb{C}$;
- *relation between multiplication and involution*: $(ab)^* = b^* a^*$ for each $a, b \in V$.

A *C*-algebra* \mathcal{A} is a *-algebra endowed with a norm $\|\cdot\|$ defined on the underlying vector space such that it becomes a Banach space and the following properties hold:

- *submultiplicativity*: $\|ab\| \leq \|a\| \|b\|$ for each $a, b \in V$;
- *the involution is an isometry*: $\|a^*\| = \|a\|$ for each $a \in V$;
- *C*-property*: $\|a^* a\| = \|a\|^2$ for each $a \in V$.

In the following we will always write $a \in \mathcal{A}$ when we consider an element of the algebra \mathcal{A} . This means that we are considering the element a of the underlying \mathbb{C} -vector space V , that in turn is the element a of the set on which the \mathbb{C} -vector structure is defined giving rise to V .

A very important example (at least in the context of quantum field theory) of C*-algebra is provided by the space $\mathcal{B}(\mathcal{H})$ of linear and continuous operators on a

Hilbert space \mathcal{H} with the composition of the operators as multiplication and the assignment of the adjoint as involution.

We find it useful to define subalgebras of given algebras.

Definition 1.4.2. Consider an algebra \mathcal{A} . A *subalgebra* \mathcal{S} of \mathcal{A} is a subspace W of the vector space V underlying \mathcal{A} that is closed with respect to the multiplication of \mathcal{A} so that the multiplication of \mathcal{A} restricted to W becomes an associative \mathbb{C} -bilinear internal operation on W giving rise to the algebra \mathcal{S} .

If \mathcal{A} is also a $*$ -algebra, we say that \mathcal{S} is a *sub- $*$ -algebra* of \mathcal{A} if it is a subalgebra of \mathcal{A} and its underlying vector space W is closed with respect to the involution of \mathcal{A} so that it can be endowed with the involution of \mathcal{A} restricted to W thus becoming a $*$ -algebra itself.

Finally if \mathcal{A} is a C^* -algebra, we say that \mathcal{S} is a *sub- C^* -algebra* of \mathcal{A} if it is a sub- $*$ -algebra of \mathcal{A} and its underlying vector space is a closed subspace of the Banach space underlying \mathcal{A} so that \mathcal{S} becomes a C^* -algebra in its own right when endowed with the norm of \mathcal{A} .

Notice that in each of the cases seen above a subalgebra of an algebra \mathcal{A} is itself an algebra constituted by a subspace of the vector space underlying \mathcal{A} which is closed with respect to all the operations that can be performed in \mathcal{A} and which is endowed with the restrictions of all the structures defined on \mathcal{A} .

Remark 1.4.3. We can obtain the smallest subalgebra \mathcal{S} (of a desired type) including a subset S of an algebra \mathcal{A} (of that type) simply taking the intersection of all the subalgebras of \mathcal{A} (of that type) that include S . In such situation we call *set of generators* the chosen subset S and *generated subalgebra* the subalgebra \mathcal{S} that we have obtained. Indeed it can happen that S is such that $\mathcal{S} = \mathcal{A}$.

It will be important for us to consider C^* -algebras that contain particular elements called unities.

Definition 1.4.4. An element 1 of an algebra \mathcal{A} is called a *unit* of \mathcal{A} if $1a = a = a1$ for each $a \in \mathcal{A}$. Each algebra possessing a unit is said to be *unital*.

Remark 1.4.5. Notice that each algebra \mathcal{A} has at most one unit. This is immediately seen assuming that both 1 and $1'$ are units of \mathcal{A} because from this assumption it follows that $1 = 11' = 1'$.

Moreover in each $*$ -algebra $1^* = 1$ since for each $a \in \mathcal{A}$ it holds that

$$1^*a = (1^*a)^{**} = (a^*1^{**})^* = (a^*1)^* = a^{**} = a$$

and similarly $a1^* = a$. Then 1^* is a unit of \mathcal{A} and uniqueness of units implies $1^* = 1$.

The last observation concerning units that we make is related to their norm: if 1 denotes the unique unit of a C^* -algebra \mathcal{A} whose underlying vector space is not trivial, we have $\|1\| = 1$. To see how this works we consider the C^* -property and

we remember that the involution is an isometry, hence $\|1\|^2 = \|1^*1\| = \|1^*\| = \|1\|$. The last equation implies that $\|1\|$ is either 0 or 1. In the first case we have $1 = 0$. 0 must be the only element of \mathcal{A} in order to be a unit of \mathcal{A} . This contradicts the hypothesis, therefore it must be $\|1\| = 1$.

Now we define maps between algebras that are compatible with the structures defined on such algebras.

Definition 1.4.6. Let \mathcal{A} and \mathcal{B} be two algebras. A map $H : \mathcal{A} \rightarrow \mathcal{B}$ is an *algebraic homomorphism* if it is compatible with the vector structures and multiplications of \mathcal{A} and \mathcal{B} , i.e. for each $a, b \in \mathcal{A}$ and each $\eta, \xi \in \mathbb{C}$ the following conditions hold:

$$\begin{aligned} H(\eta a + \xi b) &= \eta H a + \xi H b, \\ H(ab) &= (H a)(H b), \end{aligned}$$

where the first equation involves the \mathcal{A} -vector structure on the LHS and the \mathcal{B} -vector structure on the RHS, while the second equation involves the \mathcal{A} -multiplication on the LHS and the \mathcal{B} -multiplication on the RHS. A map $I : \mathcal{A} \rightarrow \mathcal{B}$ is an *algebraic isomorphism* if it is a bijective algebraic homomorphism (its inverse is automatically an algebraic homomorphism and hence an algebraic isomorphism). A map $I : \mathcal{A} \rightarrow \mathcal{A}$ is an *algebraic automorphism* if it is an algebraic isomorphism.

If \mathcal{A} and \mathcal{B} are also $*$ -algebras, a map $H : \mathcal{A} \rightarrow \mathcal{B}$ is a *$*$ -homomorphism* if it is an algebraic homomorphism compatible with the involutions of both \mathcal{A} and \mathcal{B} , i.e. $H(a^*) = (H a)^*$ for each $a \in \mathcal{A}$, where the LHS involves the \mathcal{A} -involution and the RHS involves the \mathcal{B} -involution. A map $I : \mathcal{A} \rightarrow \mathcal{B}$ is a *$*$ -isomorphism* if it is a bijective $*$ -homomorphism (its inverse is automatically a $*$ -homomorphism and hence a $*$ -isomorphism). A map $I : \mathcal{A} \rightarrow \mathcal{A}$ is a *$*$ -automorphism* if it is a $*$ -isomorphism.

The upcoming proposition provides a condition that ensures continuity for $*$ -homomorphisms between unital C^* -algebras.

Proposition 1.4.7. Let \mathcal{A} and \mathcal{B} be unital C^* -algebras and consider a $*$ -homomorphism $H : \mathcal{A} \rightarrow \mathcal{B}$. Then if H is unit preserving, for each $a \in \mathcal{A}$ we have $\|H(a)\| \leq \|a\|$, in particular H can be seen as a linear and continuous operator between the Banach spaces \mathcal{A} and \mathcal{B} with operator norm $\|H\| \leq 1$. If H is also injective, for each $a \in \mathcal{A}$ we have that $\|H(a)\| = \|a\|$, in particular H can be seen as an isometry between the Banach spaces \mathcal{A} and \mathcal{B} .

Remark 1.4.8. As a particular case of this proposition, we consider a surjective $*$ -homomorphism H from a unital C^* -algebras \mathcal{A} to a C^* -algebra \mathcal{B} . We notice that

$H1_{\mathcal{A}} \in \mathcal{B}$ and that for each $b \in \mathcal{B}$ it holds that

$$\begin{aligned}(H1_{\mathcal{A}})b &= (H1_{\mathcal{A}})(Ha) = H(1_{\mathcal{A}}a) = Ha = b, \\ b(H1_{\mathcal{A}}) &= (Ha)(H1_{\mathcal{A}}) = H(a1_{\mathcal{A}}) = Ha = b,\end{aligned}$$

where $a \in \mathcal{A}$ is such that $Ha = b$ (a exists as a consequence of the surjectivity of H). Then we recognize $H1_{\mathcal{A}}$ to be the unique unit of \mathcal{B} . Hence as a matter of fact \mathcal{B} is a unital C^* -algebra and H is unit preserving so that we can apply the last proposition. We conclude that H can be seen as a continuous linear operator between the Banach spaces \mathcal{A} and \mathcal{B} with operator norm $\|H\| \leq 1$. If H is also a $*$ -isomorphism, due to the additional hypothesis of injectivity, H becomes an isometric isomorphism between the Banach spaces \mathcal{A} and \mathcal{B} .

The next step in our path towards the construction of a quantum field theory for a bosonic field is the introduction of Weyl systems. Before we do that, we need to define symplectic spaces and symplectic maps.

Definition 1.4.9. Let V be a real vector space. We call (*non degenerate*) *symplectic form* each map $\sigma : V \times V \rightarrow \mathbb{R}$ that satisfies the following conditions:

- bilinearity: for each $u, v, w \in V$ and each $\eta, \xi \in \mathbb{R}$ it holds that

$$\begin{aligned}\sigma(\eta u + \xi v, w) &= \eta\sigma(u, w) + \xi\sigma(v, w), \\ \sigma(u, \eta v + \xi w) &= \eta\sigma(u, v) + \xi\sigma(u, w);\end{aligned}$$

- antisymmetry: $\sigma(u, v) = -\sigma(v, u)$ for each $u, v \in V$;
- non degeneracy: if $u \in V$ is such that $\sigma(u, v) = 0$ for each $v \in V$ then $u = 0$.

A *symplectic space* is a pair (V, σ) , where V is a real vector space and σ is a symplectic form on V .

Given two symplectic spaces (V, σ) and (W, ω) , we say that $s : V \rightarrow W$ is a *symplectic map* if it is linear and it is compatible with the symplectic forms σ and ω , i.e. $\omega(su, sv) = \sigma(u, v)$ for each $u, v \in V$.

Remark 1.4.10. Note that each symplectic map s between two arbitrary symplectic spaces (V, σ) and (W, ω) is injective. We can see this taking $u \in V$ such that $su = 0$ and showing that $u = 0$. Indeed this is true because $\sigma(u, v) = \omega(su, sv) = 0$ for each $v \in V$ and σ is non degenerate.

Now that we know what a symplectic space is, we are ready to define Weyl systems.

Definition 1.4.11. Let (V, σ) be a symplectic space. A *Weyl system associated to* (V, σ) is a pair (\mathcal{W}, W) where \mathcal{W} is a unital C^* -algebra and $W : V \rightarrow \mathcal{W}$ is a Weyl

map, i.e. a map that fulfils the following requirements for each $u, v \in V$:

$$\begin{aligned} W(0) &= 1, \\ W(-u) &= W(u)^*, \\ W(u)W(v) &= e^{-\frac{i}{2}\sigma(u,v)}W(u+v). \end{aligned}$$

Remark 1.4.12. The three requirements that each Weyl map W must satisfy entail that $W(u)^*W(u) = 1 = W(u)W(u)^*$. We show for example the first equality, the proof of the other being almost identical. We proceed in the following way. In the first step we exploit the second requirement, in the second step we exploit the third requirement keeping in mind that each symplectic form is antisymmetric and in the third and last step we apply the last requirement:

$$W(u)^*W(u) = W(-u)W(u) = W(0) = 1.$$

For a concrete example of Weyl system associated to an arbitrary symplectic space we refer the reader to [BGP07, Ex. 4.2.2, p. 116]. Such example shows that there exists at least one Weyl system for each symplectic space.

The requirements that define the Weyl map are such that they reproduce the canonical commutation relations of bosonic quantum fields in an exponentiated form, thus eliminating all the potential mathematical complications that can arise when we try to construct a quantum field theory starting from an algebra of bosonic fields satisfying the canonical commutation relations in their original form. This is the reason why we are interested in Weyl systems. To be more precise we are interested in a particular class of Weyl systems that we are going to define.

Definition 1.4.13. Let (V, σ) be a symplectic space. A *CCR representation* of (V, σ) is a Weyl system (\mathcal{W}, W) associated to (V, σ) such that $W(V)$ is a set of generators for the unital C^* -algebra \mathcal{W} . In such situation \mathcal{W} is called *CCR algebra*.

Once we are given a Weyl system (\mathcal{W}, W) associated to a symplectic space (V, σ) , we can easily find a CCR representation of (V, σ) considering the Weyl system associated to (V, σ) consisting of the sub- C^* -algebra generated by $W(V)$ and the Weyl map W .

This construction ensures that the existence of a CCR representation for each symplectic space is a consequence of the existence of a Weyl system for that symplectic space. The next proposition states uniqueness for CCR representations of symplectic spaces in an appropriate sense.

Proposition 1.4.14. Let (V, σ) be a symplectic space and consider two CCR representations (\mathcal{W}_1, W_1) and (\mathcal{W}_2, W_2) of (V, σ) . Then there exists a unique $*$ -isomorphism $I : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ such that $I \circ W_1 = W_2$.

Since \mathcal{W}_1 is a unital C^* -algebra, we can apply Remark 1.4.8 and conclude that I is actually unit preserving and can be interpreted as an isometric isomorphism between the Banach spaces \mathcal{W}_1 and \mathcal{W}_2 . This proposition implies that a CCR representation associated to some symplectic space is unique up to $*$ -isomorphisms.

We conclude this section with two propositions that will be essential when we will try to build quantum field theories in the next chapters.

Proposition 1.4.15. *Let \mathcal{W} be a CCR algebra. Then each unit preserving $*$ -homomorphism from \mathcal{W} to a unital C^* -algebra \mathcal{A} is injective.*

We obtain a particular case of this proposition applying Proposition 1.4.7 to \mathcal{W} : Each unit preserving $*$ -homomorphism from \mathcal{W} to a unital C^* -algebra \mathcal{A} is injective and can be seen as an isometry between the Banach spaces \mathcal{W} and \mathcal{A} .

Proposition 1.4.16. *Let (V, σ) and (W, ρ) be two symplectic spaces and let $s : V \rightarrow W$ be a symplectic map. Denoting with (\mathcal{V}, V) and (\mathcal{W}, W) two CCR representations of (V, σ) and respectively of (W, ρ) , we have that there exists a unique injective $*$ -homomorphism $H : \mathcal{V} \rightarrow \mathcal{W}$ such that $H \circ V = W \circ s$.*

We want to stress that the $*$ -homomorphism H provided by the theorem is automatically unit preserving because

$$H(1_{\mathcal{V}}) = H(V(0)) = W(s(0)) = W(0) = 1_{\mathcal{W}}.$$

Since \mathcal{V} and \mathcal{W} are both unital C^* -algebras, applying Proposition 1.4.7, we deduce that H is also an isometry between the Banach spaces \mathcal{V} and \mathcal{W} .

1.4.2 States and representations

In this subsection we focus on states and representations for a given C^* -algebra. In particular we show that a given state on each C^* -algebra induces a representation of the C^* -algebra itself on some Hilbert space. A more detailed discussion on this topic can be found in [BB09, Sect. 1.4].

We start defining states on a C^* -algebra.

Definition 1.4.17. Let \mathcal{A} be a C^* -algebra. We call *linear functional on \mathcal{A}* each $\tau : \mathcal{A} \rightarrow \mathbb{C}$ that is linear and continuous. The norm of a linear functional τ on \mathcal{A} is defined by the following formula:

$$\|\tau\| = \sup_{a \in \mathcal{A} \setminus \{0\}} \frac{\tau(a)}{\|a\|}.$$

We say that τ is *positive* if $\tau(a^*a) \geq 0$ for each $a \in \mathcal{A}$.

A *state* τ on \mathcal{A} is a positive linear functional with norm 1, i.e. $\|\tau\| = 1$. We denote the set of states on \mathcal{A} with $\mathbf{sts}\mathcal{A}$.

One of the most common examples of state is the following. Consider the C^* -algebra $\mathcal{B}(\mathcal{H})$ of linear and continuous operators on a Hilbert space \mathcal{H} and let Ω denote a norm 1 element of \mathcal{H} . Then for each $L \in \mathcal{B}(\mathcal{H})$ a state is provided by the following map

$$\begin{aligned}\tau_\Omega : \mathcal{B}(\mathcal{H}) &\rightarrow \mathbb{C} \\ L &\mapsto (\Omega, L\Omega)_{\mathcal{H}}\end{aligned}$$

where $(\cdot, \cdot)_{\mathcal{H}}$ denotes the scalar product of \mathcal{H} .

Positive linear functionals on a C^* -algebra enjoy several properties (especially if the C^* -algebra is unital). We present some of these properties in the following proposition.

Proposition 1.4.18. *Let \mathcal{A} be a C^* -algebra and let τ be a positive linear functional on \mathcal{A} . Then the following conditions hold:*

- the map

$$\begin{aligned}\mathcal{A} \times \mathcal{A} &\rightarrow \mathbb{C} \\ (a, b) &\mapsto \tau(a^*b)\end{aligned}$$

is a positive semidefinite Hermitian sesquilinear form on \mathcal{A} ;

- the Cauchy-Schwarz inequality holds for this form, i.e. for each $a, b \in \mathcal{A}$ we have

$$|\tau(a^*b)|^2 \leq \tau(a^*a) \tau(b^*b);$$

- for each $a \in \mathcal{A}$, $\tau(a^*a) = 0$ if and only if $\tau(ba) = 0$ for each $b \in \mathcal{A}$.

If \mathcal{A} possesses a unit 1 then we have some other properties:

- $\tau(a^*) = \overline{\tau(a)}$ for each $a \in \mathcal{A}$;
- $\tau(1) = \|\tau\|$.

Proof. We immediately realize that the form in the first condition of the statement is sesquilinear (antilinear in the first argument and linear in the second) and positive semidefinite. The only complication comes when we want to check that it is also Hermitian. To prove this fact, fix $a, b \in \mathcal{A}$ and $\eta \in \mathbb{C}$ and define $c = \eta a + b$. Then we find that

$$0 \leq \tau(c^*c) = |\eta|^2 \tau(a^*a) + \bar{\eta} \tau(a^*b) + \eta \tau(b^*a) + \tau(b^*b)$$

and we deduce that $\bar{\eta} \tau(a^*b) + \eta \tau(b^*a)$ must be real for each $\eta \in \mathbb{C}$. This condition for $\eta = 1$ and $\eta = i$ implies that the form is actually Hermitian.

The Cauchy-Schwarz inequality is satisfied by each positive semidefinite Hermitian sesquilinear form. Anyway we show how to proceed in this case since part of the proof has already been done. In fact the last equation implies also the Cauchy-Schwarz inequality: If $\tau(a^*a) = 0$, then $2\Re(\bar{\eta}\tau(a^*b)) + \tau(b^*b)$ must be non negative for each $\eta \in \mathbb{C}$ and hence $\tau(a^*b)$ must be zero, otherwise we can make the choice

$$\eta = -\frac{\tau(a^*b)}{\tau(a^*a)}.$$

In both cases we conclude that the Cauchy-Schwarz inequality holds.

If we take $a \in \mathcal{A}$ such that $\tau(a^*a) = 0$, from the Cauchy-Schwarz inequality it follows that

$$|\tau(ba)|^2 \leq \tau(bb^*)\tau(a^*a) = 0$$

for each $b \in \mathcal{A}$. Then $\tau(ba) = 0$. The converse implication is trivial.

Now we suppose that \mathcal{A} has a unit. The first property easily follows from hermiticity:

$$\tau(a^*) = \tau(a^*1) = \overline{\tau(1^*a)} = \overline{\tau(a)}.$$

For the second property we proceed in the following way: For each $a \in \mathcal{A}$ we find

$$|\tau(a)|^2 = |\tau(1^*a)|^2 \leq \tau(1^*1)\tau(a^*a) \leq \tau(1)\|\tau\|\|a^*a\| = \tau(1)\|\tau\|\|a\|^2;$$

if $\tau = 0$ then $\tau(1) = 0 = \|\tau\|$, otherwise we deduce $\|\tau\| \leq \tau(1)$ and then the thesis follows bearing in mind that $\tau(1) \leq \|\tau\|\|1\| = \|\tau\|$. \square

If we are dealing with states, we have some other properties that will be very helpful when we will try to find a representation for each C^* -algebra with the assignment of a state.

Proposition 1.4.19. *Let \mathcal{A} be a unital C^* -algebra and let τ be a state on \mathcal{A} . Then the following properties hold:*

- $|\tau(a)|^2 \leq \tau(a^*a)$ for each $a \in \mathcal{A}$;
- for each $a, b \in \mathcal{A}$ we have $\tau(b^*a^*ab) \leq \|a\|^2\tau(b^*b)$.

Proof. We start from the first point. For each $a \in \mathcal{A}$, using Proposition 1.4.18, we obtain

$$|\tau(a)|^2 = |\tau(1^*a)|^2 \leq \tau(1^*1)\tau(a^*a) = \tau(1)\tau(a^*a) = \tau(a^*a).$$

For the second point we fix $a, b \in \mathcal{A}$. Consider the case $\tau(b^*b) = 0$. From the first statement it follows that $\tau(cb) = 0$ for each $c \in \mathcal{A}$ and, choosing $c = b^*a^*a$, we

obtain $\tau(b^*a^*ab) = 0$ so that the thesis holds in such case. Secondly we consider the case $\tau(b^*b) > 0$ and we define the map $\rho : \mathcal{A} \rightarrow \mathbb{C}$ by setting

$$\rho(a) = \frac{\tau(b^*ab)}{\tau(b^*b)}.$$

ρ is immediately recognized as a positive linear functional on \mathcal{A} and, applying Proposition 1.4.18, we deduce that $\|\rho\| = \rho(1) = 1$, hence ρ is also a state. Then we conclude that $\rho(a^*a) \leq \|a^*a\| = \|a\|^2$, that is exactly our thesis. \square

We have discussed states in sufficient detail for our scope. It is time to turn our attention to representations of C^* -algebras on Hilbert spaces.

Definition 1.4.20. Let \mathcal{A} be a C^* -algebra and let \mathcal{H} be a Hilbert space. A *representation* of \mathcal{A} on \mathcal{H} is a $*$ -homomorphism π from \mathcal{A} to the C^* -algebra $\mathcal{B}(\mathcal{H})$ of linear and continuous operators on \mathcal{H} . Such a representation is said to be *faithful* if π is injective.

Let π be a representation of the C^* -algebra \mathcal{A} on the Hilbert space \mathcal{H} . We say that a subset S of \mathcal{H} is *invariant under \mathcal{A}* if the following condition holds:

$$\pi(\mathcal{A})S = \{\pi(a)v : a \in \mathcal{A}, v \in S\} \subseteq S.$$

We say that π is *irreducible* if the only invariant closed subspaces of \mathcal{H} are $\{0\}$ and \mathcal{H} itself.

Moreover two representations π_1 and π_2 of \mathcal{A} on the Hilbert spaces \mathcal{H}_1 and respectively \mathcal{H}_2 are said to be *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that $U \circ \pi_1(a) = \pi_2(a) \circ U$ for each $a \in \mathcal{A}$.

Before we state the main theorem of this subsection, we still need to define another ingredient.

Definition 1.4.21. Let \mathcal{A} be a C^* -algebra, let \mathcal{H} be a vector space and let $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ be a representation. A vector $\Omega \in \mathcal{H}$ is said to be *cyclic* for the representation π if $\pi(\mathcal{A})\Omega = \{\pi(a)\Omega : a \in \mathcal{A}\}$ is a dense subspace of \mathcal{H} .

We are ready to state the main theorem of this subsection.

Theorem 1.4.22. Let \mathcal{A} be a unital C^* -algebra and let τ be a state on \mathcal{A} . Then there exists a triple $(\mathcal{H}, \pi, \Omega)$, where \mathcal{H} is a Hilbert space with scalar product denoted by (\cdot, \cdot) , $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a unit preserving continuous representation of \mathcal{A} on \mathcal{H} and $\Omega \in \mathcal{H}$ is a unit cyclic vector for the representation π such that for each $a \in \mathcal{A}$ it holds that

$$(\Omega, \pi(a)\Omega) = \tau(a).$$

This triple $(\mathcal{H}, \pi, \Omega)$ is unique (up to unitary equivalence) and it is called the *GNS triple* for \mathcal{A} induced by τ .

Proof. In Proposition 1.4.18 we have seen that τ defines a positive semidefinite Hermitian sesquilinear product on \mathcal{A} . If we consider the set

$$\mathcal{N} = \{a \in \mathcal{A} : \tau(a^*a) = 0\},$$

we easily realize that this is a closed vector subspace of \mathcal{A} applying the first part of Proposition 1.4.18. Then we can consider the quotient

$$\mathcal{A}_\bullet = \frac{\mathcal{A}}{\mathcal{N}},$$

which becomes a Banach space when endowed with the quotient norm $\|\cdot\|_\bullet$ defined by the formula

$$\|a_\bullet\|_\bullet = \inf_{a \in a_\bullet} \|a\|, \quad a_\bullet \in \mathcal{A}_\bullet.$$

Consider now two equivalence classes $a_\bullet, b_\bullet \in \mathcal{A}_\bullet$ and choose $a, a' \in a_\bullet$ and $b, b' \in b_\bullet$. With this choice of representatives we evaluate $\tau(a'^*b')$. For convenience we define $n_a = a' - a$ and $n_b = b' - b$ and we immediately realize that $n_a, n_b \in \mathcal{N}$. We find that

$$\tau(a'^*b') = \tau(a^*b) + \tau(a^*n_b) + \tau(n_a^*b) + \tau(n_a^*n_b)$$

and, applying again Proposition 1.4.18, we deduce

$$\tau(a'^*b') = \tau(a^*b)$$

because $\tau(a^*n_b) = 0$, $\tau(n_a^*b) = \overline{\tau(b^*n_a)} = 0$ and $\tau(n_a^*n_b) = 0$. This shows that the map

$$\begin{aligned} (\cdot, \cdot)_\bullet : \mathcal{A}_\bullet \times \mathcal{A}_\bullet &\rightarrow \mathbb{C} \\ (a_\bullet, b_\bullet) &\mapsto \tau(a^*b), \quad a \in a_\bullet, b \in b_\bullet \end{aligned}$$

is well defined. It is immediate to check that it is a positive semidefinite Hermitian sesquilinear form. Now we show that it is also positive definite. Consider $a_\bullet \in \mathcal{A}_\bullet$ such that $(a_\bullet, a_\bullet)_\bullet = 0$. By definition of $(\cdot, \cdot)_\bullet$, this means that we have $\tau(a^*a) = 0$ for each $a \in a_\bullet$. Then a_\bullet coincides with \mathcal{N} , that is the zero element of \mathcal{A}_\bullet . We conclude that $(\cdot, \cdot)_\bullet$ is a scalar product on \mathcal{A}_\bullet , so that \mathcal{A}_\bullet becomes a pre-Hilbert space when endowed with $(\cdot, \cdot)_\bullet$. This can be completed and we obtain an Hilbert space \mathcal{H} . We denote its scalar product with (\cdot, \cdot) and we remind the reader that the pre-Hilbert space \mathcal{A}_\bullet is isometrically isomorphic to a certain subspace \mathcal{S} of \mathcal{H} , therefore the composition of the inclusion map of \mathcal{S} in \mathcal{H} with the isometrical isomorphism J from \mathcal{A}_\bullet to \mathcal{S} is an isometry. We denote this isometry with I .

Consider now $a \in \mathcal{A}$ and $b_\bullet \in \mathcal{A}_\bullet$ and choose two representatives $b, b' \in b_\bullet$. For convenience we define $n = b' - b$ and we notice that $n \in \mathcal{N}$. From Proposition 1.4.19 we deduce that $\tau(n^*a^*an) = 0$. Then an falls in \mathcal{N} and we conclude that

$[ab]_{\bullet} = [ab']_{\bullet}$. This shows that for each $a \in \mathcal{A}$, the map

$$\begin{aligned} L_a : \mathcal{A}_{\bullet} &\rightarrow \mathcal{A}_{\bullet} \\ b_{\bullet} &\mapsto [ab]_{\bullet}, \quad b \in b_{\bullet} \end{aligned}$$

is well defined. L_a is also linear, as one immediately recognizes. For each $a \in \mathcal{A}$ we show that L_a is also continuous on \mathcal{A}_{\bullet} endowed with the norm induced by $(\cdot, \cdot)_{\bullet}$. We fix $a \in \mathcal{A}$ for each $b_{\bullet} \in \mathcal{A}_{\bullet}$ and we apply again the last part of Proposition 1.4.19. Then we find

$$(L_a b_{\bullet}, L_a b_{\bullet})_{\bullet} = ([ab]_{\bullet}, [ab]_{\bullet})_{\bullet} = \tau(b^* a^* ab) \leq \|a\|^2 \tau(b^* b) = \|a\|^2 (b_{\bullet}, b_{\bullet})_{\bullet}.$$

This means exactly the continuity of L_a with respect to the norm induced by $(\cdot, \cdot)_{\bullet}$. Moreover its norm as a linear and continuous operator on \mathcal{A}_{\bullet} is controlled from above by $\|a\|$: we write that $\|L_a\| \leq \|a\|$.

Recalling the isometry $I : \mathcal{A}_{\bullet} \rightarrow \mathcal{H}$ and the isometrical isomorphism $J : \mathcal{A}_{\bullet} \rightarrow \mathcal{S}$, for each $a \in \mathcal{A}$ we define $L'_a = I \circ L_a \circ J^{-1}$. We have that L'_a is a linear and continuous operator from \mathcal{S} to \mathcal{H} with norm $\|L'_a\| \leq \|a\|$. Since \mathcal{H} is complete, we can find a unique linear and continuous extension of L'_a defined on the closure of \mathcal{S} , i.e. \mathcal{H} . We denote such linear and continuous operator on \mathcal{H} with $\pi(a)$ and we find that $\|\pi(a)\| = \|L'_a\| \leq \|a\|$. In this way we the map $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is automatically defined. π is linear as the reader can directly check from its definition. Moreover $\|\pi(a)\| \leq \|a\|$ shows that π is continuous. We must only check that for each $a, b \in \mathcal{A}$ the following equations hold:

$$\begin{aligned} \pi(ab) &= \pi(a) \pi(b), \quad \forall a, b \in \mathcal{A}; \\ \pi(a^*) &= \pi(a)^*, \quad \forall a \in \mathcal{A}. \end{aligned}$$

and then we have a continuous representation of \mathcal{A} on \mathcal{H} . Fix $a \in \mathcal{A}$ and $v \in \mathcal{H}$. To simplify the inspection of these equations we give an expression of $\pi(a)v$. From the definition of $\pi(a)$ as the unique linear and continuous extension of L'_a , we find a Cauchy sequence $\{v_n\} \subseteq \mathcal{S}$ that converges to $v \in \mathcal{H}$ such that $\{L'_a v_n\}$ converges to $\pi(a)v$ in \mathcal{H} and for each n we choose a representative v'_n of the equivalence class $J^{-1}v_n \in \mathcal{A}_{\bullet}$. Therefore we have

$$\pi(a)v = \lim_{n \rightarrow \infty} (L'_a v_n) = \lim_{n \rightarrow \infty} ((I \circ L_a \circ J^{-1})v_n) = \lim_{n \rightarrow \infty} (I[av'_n]_{\bullet}). \quad (1.4.1)$$

This formula allows us to easily check the first equation above. For the second equation we must also keep in mind that the involution of $\mathcal{B}(\mathcal{H})$ is the map $\mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, $L \mapsto L^{\dagger}$, where L^{\dagger} is the adjoint of L with respect to the scalar

product (\cdot, \cdot) of \mathcal{H} , that is

$$(L^\dagger v, w) = (v, Lw) \quad \forall v, w \in \mathcal{H}.$$

Therefore the condition that we must actually check is the following:

$$(v, \pi(a)w) = (\pi(a^*)v, w) \quad \forall v, w \in \mathcal{H} \quad \forall a \in \mathcal{A}.$$

This is easily seen to hold using eq. (1.4.1).

Now we have to find a cyclic unit vector in \mathcal{H} that satisfies the condition of the statement. Using the unit of \mathcal{A} , we define $\Omega = I[1]_\bullet$. This is indeed an element of \mathcal{H} . We fix $a \in \mathcal{A}$ and we try to evaluate $(\Omega, \pi(a)\Omega)$. In first place we use eq. (1.4.1) with $v = \Omega$. As a consequence of the definition of Ω the formula becomes simpler:

$$\pi(a)\Omega = I[a1]_\bullet = I[a]_\bullet.$$

Hence we have that $\pi(\mathcal{A})\Omega = I(\mathcal{A})_\bullet = \mathcal{S}$ and, since \mathcal{S} is dense in \mathcal{H} , Ω is indeed cyclic. Moreover we get

$$(\Omega, \pi(a)\Omega) = (I[1]_\bullet, I[a]_\bullet) = ([1]_\bullet, [a]_\bullet)_\bullet = \tau(1^*a) = \tau(a).$$

Since $\pi(1) = \text{id}_\mathcal{H}$, as it can be checked via direct inspection, the last equation for $a = 1$ implies that $(\Omega, \Omega) = 1$.

We have built a triple $(\pi, \mathcal{H}, \Omega)$ with the properties required in the statement. To complete the proof we must show that such triple is unique up to a unitary transformation. To this end suppose that we have another triple $(\pi', \mathcal{H}', \Omega')$ of the same type and for convenience we denote with \mathcal{S} the dense subspace $\pi(\mathcal{A})\Omega$ of \mathcal{H} and with \mathcal{S}' the dense subspace $\pi'(\mathcal{A})\Omega'$ of \mathcal{H}' . If we have $a, b \in \mathcal{A}$ such that $\pi(a)\Omega = \pi(b)\Omega$, then it holds also that $\pi'(a)\Omega = \pi'(b)\Omega'$. To check this fact fix an arbitrary $v' \in \mathcal{H}$ and, using the density of \mathcal{S}' in \mathcal{H}' , choose a sequence $\{v'_n\} \subseteq \mathcal{S}'$ that converges to v' in \mathcal{H}' . By definition of \mathcal{S}' , for each n we also find $a_n \in \mathcal{A}$ such that $\pi'(a_n)\Omega' = v'_n$. Using (\cdot, \cdot) and $(\cdot, \cdot)'$ to denote the scalar products of \mathcal{H} and respectively \mathcal{H}' and bearing in mind the properties fulfilled by each of the triples, we deduce that for each $c, d \in \mathcal{A}$

$$\begin{aligned} (\pi(c)\Omega, \pi(d)\Omega) &= \left(\Omega, \pi(c)^\dagger \pi(d)\Omega \right) = (\Omega, \pi(c^*d)\Omega) = \tau(c^*d), \\ (\pi'(c)\Omega', \pi'(d)\Omega') &= \left(\Omega', \pi'(c)^\dagger \pi'(d)\Omega' \right) = (\Omega', \pi'(c^*d)\Omega') = \tau(c^*d). \end{aligned}$$

Hence we find that

$$\begin{aligned}
 (v', \pi'(a)\Omega')' &= \lim_{n \rightarrow \infty} (\pi'(a_n)\Omega', \pi'(a)\Omega') \\
 &= \lim_{n \rightarrow \infty} (\pi(a_n)\Omega, \pi(a)\Omega) \\
 &= \lim_{n \rightarrow \infty} (\pi(a_n)\Omega, \pi(b)\Omega) \\
 &= \lim_{n \rightarrow \infty} (\pi'(a_n)\Omega', \pi'(b)\Omega') \\
 &= (v', \pi'(b)\Omega')'.
 \end{aligned}$$

This holds for each $v' \in \mathcal{H}'$. Therefore the map

$$\begin{aligned}
 V : \quad \mathcal{S} &\rightarrow \mathcal{H}' \\
 \pi(a)\Omega &\mapsto \pi'(a)\Omega'
 \end{aligned}$$

is well defined. Moreover V is trivially linear and, from the considerations made above, we deduce that for each $a, b \in \mathcal{A}$ the following equation holds:

$$(V(\pi(a)\Omega), V(\pi(b)\Omega))' = (\pi(a)\Omega, \pi(b)\Omega).$$

In particular this implies that V is a linear and continuous operator from the dense subspace \mathcal{S} of the Hilbert space \mathcal{H} to the other Hilbert space \mathcal{H}' . Then there exists a unique linear and continuous extension U of V defined on the closure of \mathcal{S} , i.e. $U : \mathcal{H} \rightarrow \mathcal{H}'$. With the help of our last equation we show that U is unitary. Fix $v, w \in \mathcal{H}$. We find Cauchy sequences $\{v_n\}, \{w_n\} \subseteq \mathcal{S}$ that converge to v and respectively w in \mathcal{H} such that $\{Vv_n\}$ and $\{Vw_n\}$ converge to Uv and, respectively, Uw in \mathcal{H}' . Then for each n we find $a_n, b_n \in \mathcal{A}$ such that $\pi(a_n)\Omega = v_n$ and $\pi(b_n)\Omega = w_n$. Recalling that the scalar product is always continuous in both its arguments, we obtain:

$$\begin{aligned}
 (Uv, Uw)' &= \lim_{n \rightarrow \infty} (V(\pi(a_n)\Omega), V(\pi(b_n)\Omega))' \\
 &= \lim_{n \rightarrow \infty} (\pi(a_n)\Omega, \pi(b_n)\Omega) \\
 &= \lim_{n \rightarrow \infty} (v_n, w_n) \\
 &= (v, w).
 \end{aligned}$$

Since the last equation holds for each $v, w \in \mathcal{H}$, we deduce that U is unitary as required. The only property that must still be checked is the following:

$$U \circ \pi(a) = \pi'(a) \circ U \quad \forall a \in \mathcal{A}.$$

By construction U coincides with V on \mathcal{S} and so $U(\pi(a)\Omega) = \pi'(a)\Omega'$ for each

$a \in \mathcal{A}$. From this it follows that for each $a \in \mathcal{A}$ we have

$$(\Omega', \pi'(a) \Omega')' = \tau(a) = (\Omega, \pi(a) \Omega) = (U\Omega, U(\pi(a) \Omega))' = (U\Omega, \pi'(a) \Omega')'$$

and, since $\pi'(\mathcal{A})\Omega = \mathcal{S}'$ is dense in \mathcal{H}' and $(\cdot, \cdot)'$ is continuous in its second argument, it follows that $(\Omega', v')' = (U\Omega, v')'$ for each $v' \in \mathcal{H}'$, which is to say $\Omega' = U\Omega$. This fact provides us the equation that leads to the conclusion of the proof:

$$U(\pi(a) \Omega) = \pi'(a)(U\Omega) \quad \forall a \in \mathcal{A}.$$

As a preliminary step, we observe that for each $a, b \in \mathcal{A}$ it holds

$$\begin{aligned} U(\pi(a)(\pi(b) \Omega)) &= U(\pi(ab) \Omega) = \pi'(ab)(U\Omega) = \pi'(a)(\pi'(b)(U\Omega)) \\ &= \pi'(a)(U(\pi'(b) \Omega)). \end{aligned}$$

Consider $a \in \mathcal{A}$ and $v \in \mathcal{H}$. As usual we find a Cauchy sequence $\{v_n\} \subseteq \mathcal{S}$ that converges to v in \mathcal{H} and for each n we find $a_n \in \mathcal{A}$ such that $\pi(a_n) \Omega = v_n$. Then, reminding of the continuity of U , $\pi(a)$ and $\pi'(a)$, we have

$$U(\pi(a)v) = \lim_{n \rightarrow \infty} U(\pi(a)(\pi(a_n) \Omega)) = \lim_{n \rightarrow \infty} \pi'(a)(U(\pi(a_n) \Omega)) = \pi'(a)(Uv).$$

Since this holds for each $a \in \mathcal{A}$ and each $v \in \mathcal{H}$, the proof is complete. \square

1.5 Category theory

This section concludes the preliminary part of the thesis. We devote it to the presentation of some notions from category theory that will be extensively used in the next chapters. This is essentially due to the fact that it is possible to construct a quantum field theory as a covariant functor between appropriate categories. As a matter of fact we only need very few notions of category theory so that, despite of its brevity, the current section, unlike the previous ones, is totally self contained and sufficient for our scopes. Anyway as general reference about this topic we suggest [ML98].

We start defining what it is meant for a category.

Definition 1.5.1. A category \mathfrak{C} consists of a set of *objects* $\text{Obj}_{\mathfrak{C}}$, a set of *morphisms* $\text{Mor}_{\mathfrak{C}}(A, B)$ from A to B for each pair of objects (A, B) and a map, called *composition law*,

$$\begin{aligned} \circ : \text{Mor}_{\mathfrak{C}}(B, C) \times \text{Mor}_{\mathfrak{C}}(A, B) &\rightarrow \text{Mor}_{\mathfrak{C}}(A, C) \\ (g, f) &\mapsto g \circ f \end{aligned}$$

for each triple of objects (A, B, C) . The following axioms (we call them *category*

axioms) are assumed to hold:

- *identity law*: for each $A \in \text{Obj}_{\mathfrak{C}}$ the set $\text{Mor}_{\mathfrak{C}}(A, A)$ must contain at least an element id_A such that, for each $B \in \text{Obj}_{\mathfrak{C}}$, each $f \in \text{Mor}_{\mathfrak{C}}(A, B)$ and each $g \in \text{Mor}_{\mathfrak{C}}(B, A)$, it holds that

$$\begin{aligned} f \circ \text{id}_A &= f, \\ \text{id}_A \circ g &= g; \end{aligned}$$

- *associative law*: for each $A, B, C, D \in \text{Obj}_{\mathfrak{C}}$, each $f \in \text{Mor}_{\mathfrak{C}}(A, B)$, each $g \in \text{Mor}_{\mathfrak{C}}(B, C)$ and each $h \in \text{Mor}_{\mathfrak{C}}(C, D)$ it holds that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Let \mathfrak{C} be a category. A *subcategory* \mathfrak{S} of \mathfrak{C} is a category such that $\text{Obj}_{\mathfrak{S}} \subseteq \text{Obj}_{\mathfrak{C}}$, $\text{Mor}_{\mathfrak{S}}(A, B) \subseteq \text{Mor}_{\mathfrak{C}}(A, B)$ for each $A, B \in \text{Obj}_{\mathfrak{S}}$. Moreover we require that:

- for each object A of \mathfrak{S} the identity morphism of $\text{Mor}_{\mathfrak{S}}(A, A)$ coincides with the identity morphism of $\text{Mor}_{\mathfrak{C}}(A, A)$;
- for each $A, B, C \in \text{Obj}_{\mathfrak{S}}$, each $f \in \text{Mor}_{\mathfrak{S}}(A, B)$ and each $g \in \text{Mor}_{\mathfrak{S}}(B, C)$ the composition $g \circ f$ in \mathfrak{S} coincides with the composition $g \circ f$ in \mathfrak{C} .

We say that \mathfrak{S} is a *full subcategory* of \mathfrak{C} if it is a subcategory of \mathfrak{C} and $\text{Mor}_{\mathfrak{S}}(A, B) = \text{Mor}_{\mathfrak{C}}(A, B)$ for each $A, B \in \text{Obj}_{\mathfrak{S}}$.

Example 1.5.2. Examples of categories are:

the category whose objects are sets, whose morphisms are functions between pairs of sets and whose composition law is provided by the composition of functions;

the category whose objects are topological spaces, whose morphisms are continuous functions between pairs of topological spaces and whose composition law is provided by the composition of functions;

the category whose objects are groups, whose morphisms are homomorphisms between pairs of groups and whose composition law is provided by the composition of functions.

One easily checks the validity of the category axioms in these cases. One may even note that the second category and the third category are (non full) subcategories of the first one.

Now we define covariant and contravariant functors.

Definition 1.5.3. Let \mathfrak{A} and \mathfrak{B} be two categories. A *covariant functor* \mathcal{F} from \mathfrak{A} to \mathfrak{B} is a map

$$\mathcal{F} : \text{Obj}_{\mathfrak{A}} \rightarrow \text{Obj}_{\mathfrak{B}}$$

together with a collection of maps

$$\{\mathcal{F} : \text{Mor}_{\mathfrak{A}}(A, B) \rightarrow \text{Mor}_{\mathfrak{B}}(\mathcal{F}(A), \mathcal{F}(B)) \text{ for } A, B \in \text{Obj}_{\mathfrak{A}}\}$$

such that the following requirements, called *covariant axioms*, are fulfilled:

- the composition of morphisms is preserved, i.e. for each $A, B, C \in \text{Obj}_{\mathfrak{A}}$, each $f \in \text{Mor}_{\mathfrak{A}}(A, B)$ and each $g \in \text{Mor}_{\mathfrak{A}}(B, C)$ we have

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f),$$

where on the LHS we have the \mathfrak{A} -composition law, while on the RHS we have the \mathfrak{B} -composition law;

- the identity map of an object A of \mathfrak{A} is mapped to the identity map of the corresponding object $\mathcal{F}(A)$ of \mathfrak{B} , i.e. for each $A \in \text{Obj}_{\mathfrak{A}}$ we have

$$\mathcal{F}(\text{id}_A) = \text{id}_{\mathcal{F}(A)}.$$

A *contravariant functor* \mathcal{G} from \mathfrak{A} to \mathfrak{B} is a map

$$\mathcal{G} : \text{Obj}_{\mathfrak{A}} \rightarrow \text{Obj}_{\mathfrak{B}}$$

together with a collection of maps

$$\{\mathcal{G} : \text{Mor}_{\mathfrak{A}}(A, B) \rightarrow \text{Mor}_{\mathfrak{B}}(\mathcal{G}(B), \mathcal{G}(A)) \text{ for } A, B \in \text{Obj}_{\mathfrak{A}}\}$$

such that the following requirements, called *contravariant axioms*, are fulfilled:

- the composition of morphisms is reversed, i.e. for each $A, B, C \in \text{Obj}_{\mathfrak{A}}$, each $f \in \text{Mor}_{\mathfrak{A}}(A, B)$ and each $g \in \text{Mor}_{\mathfrak{A}}(B, C)$ we have

$$\mathcal{G}(g \circ f) = \mathcal{G}(f) \circ \mathcal{G}(g),$$

where on the LHS we have the \mathfrak{A} -composition law, while on the RHS we have the \mathfrak{B} -composition law;

- the identity map of an object A of \mathfrak{A} is mapped to the identity map of the corresponding object $\mathcal{G}(A)$ of \mathfrak{B} , i.e. for each $A \in \text{Obj}_{\mathfrak{A}}$ we have

$$\mathcal{G}(\text{id}_A) = \text{id}_{\mathcal{G}(A)}.$$

We sometimes denote a covariant functor \mathcal{F} from a category \mathfrak{A} to a category \mathfrak{B} with $\mathcal{F} : \mathfrak{A} \rightarrow \mathfrak{B}$ (the direction of the upper arrow denotes that the composition is preserved). On the contrary, for a contravariant functor \mathcal{G} from \mathfrak{A} to \mathfrak{B} we write

$\mathcal{F} : \mathfrak{A} \xrightarrow{\leftarrow} \mathfrak{B}$ (here the direction of the upper arrow denotes that the composition is reversed).

Example 1.5.4. We show an example of a covariant functor. Consider the category \mathbf{tsp} of topological spaces and the category \mathbf{set} of sets. We define \mathcal{F} imposing $\mathcal{F}(X) = S$ for each $X \in \mathbf{Obj}_{\mathbf{tsp}}$, where S is the underlying set of X and τ is its topology, and imposing $\mathcal{F}(f) = f$ for each $X_1, X_2 \in \mathbf{Obj}_{\mathbf{tsp}}$ and each $f \in \mathbf{Mor}_{\mathbf{tsp}}(X_1, X_2)$. It is immediate to check that \mathcal{F} satisfies the covariant axioms. Notice that covariant functors like \mathcal{F} are called *forgetful functors*, since they “forget” of some structure or property possessed by the objects and morphisms of the starting category.

Definition 1.5.5. Let \mathcal{F} be a covariant functor from a category \mathfrak{A} to a category \mathfrak{B} and let \mathcal{G} be a covariant functor from \mathfrak{B} to a category \mathfrak{C} . The composition of \mathcal{F} and \mathcal{G} is the covariant functor whose map between the objects $\mathcal{G} \circ \mathcal{F} : \mathbf{Obj}_{\mathfrak{A}} \rightarrow \mathbf{Obj}_{\mathfrak{C}}$ is the composition of the maps $\mathcal{F} : \mathbf{Obj}_{\mathfrak{A}} \rightarrow \mathbf{Obj}_{\mathfrak{B}}$ and $\mathcal{G} : \mathbf{Obj}_{\mathfrak{B}} \rightarrow \mathbf{Obj}_{\mathfrak{C}}$ and whose maps between the morphisms are defined in the following way: for each $A, B \in \mathbf{Obj}_{\mathfrak{A}}$, we obtain

$$\mathcal{G} \circ \mathcal{F} : \mathbf{Mor}_{\mathfrak{A}}(A, B) \rightarrow \mathbf{Mor}_{\mathfrak{C}}((\mathcal{G} \circ \mathcal{F})(A), (\mathcal{G} \circ \mathcal{F})(B))$$

composing the maps

$$\begin{aligned} \mathcal{F} : \mathbf{Mor}_{\mathfrak{A}}(A, B) &\rightarrow \mathbf{Mor}_{\mathfrak{B}}(\mathcal{F}(A), \mathcal{F}(B)), \\ \mathcal{G} : \mathbf{Mor}_{\mathfrak{B}}(\mathcal{F}(A), \mathcal{F}(B)) &\rightarrow \mathbf{Mor}_{\mathfrak{C}}((\mathcal{G} \circ \mathcal{F})(A), (\mathcal{G} \circ \mathcal{F})(B)). \end{aligned}$$

The composition of contravariant functors is a covariant functor defined similarly, the only difference being that we must compose the maps

$$\begin{aligned} \mathcal{F} : \mathbf{Mor}_{\mathfrak{A}}(A, B) &\rightarrow \mathbf{Mor}_{\mathfrak{B}}(\mathcal{F}(B), \mathcal{F}(A)), \\ \mathcal{G} : \mathbf{Mor}_{\mathfrak{B}}(\mathcal{F}(B), \mathcal{F}(A)) &\rightarrow \mathbf{Mor}_{\mathfrak{C}}((\mathcal{G} \circ \mathcal{F})(A), (\mathcal{G} \circ \mathcal{F})(B)) \end{aligned}$$

to obtain

$$\mathcal{G} \circ \mathcal{F} : \mathbf{Mor}_{\mathfrak{A}}(A, B) \rightarrow \mathbf{Mor}_{\mathfrak{C}}((\mathcal{G} \circ \mathcal{F})(A), (\mathcal{G} \circ \mathcal{F})(B)).$$

Finally the composition of a covariant functor with a contravariant functor (or vice versa) is the contravariant functor defined as above paying attention to the reversal in the direction of the morphisms caused by a contravariant functor.

One can easily check that the definition above is well posed and that the composed functors are actually covariant in the first two cases and contravariant in last case. The composition of functors gives us the opportunity to present a new example of category, the “category of categories”, whose objects are categories, whose

morphisms are covariant and contravariant functors and whose composition law is the composition of functors.

To conclude this section we want to introduce another notion from category theory, specifically that of natural transformation.

Definition 1.5.6. Let \mathfrak{A} and \mathfrak{B} be categories and let \mathcal{F} and \mathcal{G} be covariant functors from \mathfrak{A} to \mathfrak{B} . A *covariant natural transformation* \mathbf{n} from \mathcal{F} to \mathcal{G} is a collection of morphisms of the category \mathfrak{B}

$$\{\mathbf{n}_A \in \text{Mor}_{\mathfrak{B}}(\mathcal{F}(A), \mathcal{G}(A)) \text{ for } A \in \text{Obj}_{\mathfrak{A}}\}$$

such that the following condition, called *covariant naturality axiom*, is verified:

for each $A, B \in \text{Obj}_{\mathfrak{A}}$ and each $f \in \text{Mor}_{\mathfrak{A}}(A, B)$ we have that

$$\mathbf{n}_B \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \mathbf{n}_A.$$

Otherwise let \mathcal{F} and \mathcal{G} be contravariant functors from \mathfrak{A} to \mathfrak{B} . A *contravariant natural transformation* \mathbf{n} from \mathcal{F} to \mathcal{G} is again a collection of morphisms of the category \mathfrak{B}

$$\{\mathbf{n}_A \in \text{Mor}_{\mathfrak{B}}(\mathcal{F}(A), \mathcal{G}(A)) \text{ for } A \in \text{Obj}_{\mathfrak{A}}\}$$

such that the following condition, called *contravariant naturality axiom*, is verified:

for each $A, B \in \text{Obj}_{\mathfrak{A}}$ and each $f \in \text{Mor}_{\mathfrak{A}}(A, B)$ we have that

$$\mathbf{n}_A \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \mathbf{n}_B.$$

For each $A \in \text{Obj}_{\mathfrak{A}}$ we say that \mathbf{n}_A is the *A-component of the natural transformation* \mathbf{n} (whether \mathbf{n} is covariant or contravariant).

A *covariant (contravariant) natural isomorphism* \mathbf{i} is a covariant (respectively contravariant) natural transformation such that each of its components is an isomorphism between the appropriate objects (i.e. a bijective morphism whose inverse is a morphism).

For natural transformations we introduce a notation (similar to the one introduced for functors) that allows us to easily distinguish the covariant case from the contravariant one: a covariant natural transformation \mathbf{n} from $\mathcal{F} : \mathfrak{A} \rightarrow \mathfrak{B}$ to $\mathcal{G} : \mathfrak{A} \rightarrow \mathfrak{B}$ will be denoted by $\mathbf{n} : \mathcal{F} \rightarrow \mathcal{G}$, whereas a contravariant natural transformation \mathbf{m} from $\mathcal{F} : \mathfrak{A} \leftarrow \mathfrak{B}$ to $\mathcal{G} : \mathfrak{A} \leftarrow \mathfrak{B}$ will be denoted by $\mathbf{m} : \mathcal{F} \leftarrow \mathcal{G}$.

Chapter 2

The generally covariant locality principle

This chapter is divided in three sections. In the first one, following [BFV03], we present an approach to quantum field theory on curved spacetimes known as *generally covariant locality principle* (abbreviated by the acronym *GCLP*) and we study the properties of *locally covariant quantum field theories* (or *LCQFT*), that are quantum field theories formulated following the scheme provided by the GCLP. Our main goal is to show that this family of quantum field theories automatically satisfies the Haag-Kastler axioms, originally stated in [HK64]. Hence on the one hand the GCLP recovers exactly the algebraic approach to quantum field theory suggested by Haag and Kastler, while on the other hand it has the advantage of emphasizing the common features of the quantization procedures on different spacetimes and elegantly accounts for the covariance property required by general relativity for any theory to be physical.

In the second section we show how a LCQFT can be constructed starting from the Cauchy problem for a classical field over a globally hyperbolic spacetime. Here we follow an approach similar to that in [BFV03, Sect. 4.3].

We conclude this chapter showing some examples of concrete locally covariant quantum field theories. Specifically we study the cases of the Klein-Gordon field, of the Proca field and of the electromagnetic field.

2.1 Locally covariant quantum field theory

Locally covariant quantum field theories are defined in terms of covariant functors between appropriate categories. The first part of this section is devoted to a detailed presentation of such categories.

2.1.1 The categories \mathbf{ghs} and \mathbf{alg}

We start defining both \mathbf{ghs} and \mathbf{alg} . In the subsequent remarks we study in detail some properties of their morphisms and then we check that they actually satisfy the category axioms stated in Definition 1.5.1.

Definition 2.1.1. The category \mathbf{ghs} is defined in the following way:

- Objects are d -dimensional globally hyperbolic spacetimes $\mathcal{M} = (M, g, \mathbf{o}, \mathbf{t})$;
- The set of morphism $\mathbf{Mor}_{\mathbf{ghs}}(\mathcal{M}, \mathcal{N})$ between the objects $\mathcal{M} = (M, g, \mathbf{o}, \mathbf{t})$ and $\mathcal{N} = (N, h, \mathbf{p}, \mathbf{u})$ encompasses all the orientation $(\psi'_*\mathbf{o} = \mathbf{p}|_{\psi(M)})$ and time orientation $(\psi'_*\mathbf{t} = \mathbf{u}|_{\psi(M)})$ preserving isometric embeddings $\psi : \mathcal{M} \rightarrow \mathcal{N}$ whose images $\psi(M)$ are \mathcal{N} -causally convex open subsets of N ;
- The composition law is provided by the usual composition of functions.

\mathbf{alg} is the category whose objects are unital C^* -algebras, whose set of morphisms $\mathbf{Mor}_{\mathbf{alg}}(\mathcal{A}, \mathcal{B})$ between the objects \mathcal{A} and \mathcal{B} comprises all the injective unit preserving $*$ -homomorphisms $H : \mathcal{A} \rightarrow \mathcal{B}$ and whose composition law is again the usual composition of functions.

Before the check of the category axioms for \mathbf{ghs} and \mathbf{alg} , we devote few lines to some comments on their morphisms.

Remark 2.1.2. Dealing with \mathbf{ghs} , consider $\mathcal{M} = (M, g, \mathbf{o}, \mathbf{t})$, $\mathcal{N} = (N, h, \mathbf{p}, \mathbf{u}) \in \mathbf{Obj}_{\mathbf{ghs}}$ and $\psi \in \mathbf{Mor}_{\mathbf{ghs}}(\mathcal{M}, \mathcal{N})$. We have that $\psi(M)$ is a \mathcal{N} -causally convex open subset of N . It is also connected because it is the image through ψ of M , which is connected being a manifold. Then, recalling Remark 1.2.11, we can consider the oriented and time oriented Lorentzian manifold $\mathcal{N}|_{\psi(M)} = (\psi(M), h|_{\psi(M)}, \mathbf{p}|_{\psi(M)}, \mathbf{u}|_{\psi(M)})$. If we consider the diffeomorphism $\psi' : M \rightarrow \psi(M)$ (see the end of Remark 1.1.7) and we recall that ψ is isometric and preserves orientation and time orientation, we can introduce on $\psi(M)$ the (fiberwise) symmetric and (fiberwise) non degenerate section of $T^{(0,2)}\psi(M)$ $\psi'_*g = h|_{\psi(M)}$, the set of d -forms $\psi'_*\mathbf{o} = \mathbf{p}|_{\psi(M)}$ and the vector field $\psi'_*\mathbf{t} = \mathbf{u}|_{\psi(M)}$. Hence we recognize that ψ'_*g is a Lorentzian metric on $\psi(M)$, that $\psi(M)$ is orientable and $\psi'_*\mathbf{o}$ is a choice of an orientation and that $(\psi(M), \psi'_*g)$ is a time orientable Lorentzian manifold and $\psi'_*\mathbf{t}$ is a choice of a time orientation. Therefore we can define the oriented and time oriented Lorentzian manifold $(\psi(M), \psi'_*g, \psi'_*\mathbf{o}, \psi'_*\mathbf{t})$ that we denote with $\psi(\mathcal{M})$ and it immediately turns out that $\psi(\mathcal{M}) = \mathcal{N}|_{\psi(M)}$. So we will usually write $\psi(\mathcal{M})$ in place of $\mathcal{N}|_{\psi(M)}$. There is even more: applying Proposition 1.2.16, we realize that $\psi(M)$ is an \mathcal{N} -globally hyperbolic connected open subset of N and then, applying Remark 1.2.13, we deduce that $\psi(\mathcal{M})$ is itself a d -dimensional globally hyperbolic spacetime, i.e. an object of \mathbf{ghs} in its own right, and we can easily recognize that the following two maps are actually morphisms of \mathbf{ghs} :

- ψ' becomes a bijective morphism from \mathcal{M} to $\psi(\mathcal{M})$ whose inverse ψ'^{-1} is a morphism from $\psi(\mathcal{M})$ to \mathcal{M} ;
- the inclusion map $\iota_{\psi(M)}^N$ of $\psi(M)$ into N becomes a morphism from $\psi(\mathcal{M})$ to \mathcal{N} : This is a consequence of a more general fact that holds for each object $\mathcal{O} = (O, i, \mathbf{q}, \mathbf{v})$ of \mathbf{ghs} and each \mathcal{O} -causally convex connected open subset Ω of O , specifically that the inclusion map ι_Ω^O of Ω in O is actually a morphism from $\mathcal{O}|_\Omega$ to \mathcal{O} (to check this fact note that Remark 1.1.7 implies that Ω is a submanifold of O and that the inclusion map ι_Ω^O is an embedding and apply Proposition 1.2.16 and Remark 1.2.13 to obtain the globally hyperbolic spacetime $\mathcal{O}|_\Omega = (\Omega, i|_\Omega, \mathbf{q}|_\Omega, \mathbf{v}|_\Omega)$).

Using these two facts we can decompose each $\psi \in \mathbf{Mor}_{\mathbf{ghs}}(\mathcal{M}, \mathcal{N})$ in two morphisms $\iota_{\psi(M)}^N \in \mathbf{Mor}_{\mathbf{ghs}}(\psi(\mathcal{M}), \mathcal{N})$ and $\psi' \in \mathbf{Mor}_{\mathbf{ghs}}(\mathcal{M}, \psi(\mathcal{M}))$ (which is bijective and whose inverse is a morphism from $\psi(\mathcal{M})$ to \mathcal{M}) according to the formula $\psi = \iota_{\psi(M)}^N \circ \psi'$.

Remark 2.1.3. As anticipated, we make some observations also on the morphisms of \mathbf{alg} . Recalling Proposition 1.4.7 and bearing in mind that all the objects of \mathbf{alg} are unital C^* -algebras, we see that each morphism of this category can also be seen as an isometry between the Banach spaces underlying its domain and its codomain. We can use this fact to obtain results similar to that found for the morphisms of \mathbf{ghs} . Specifically consider two objects \mathcal{A} and \mathcal{B} and a morphism $H : \mathcal{A} \rightarrow \mathcal{B}$ of \mathbf{alg} . We consider the vector spaces A and B that underlie \mathcal{A} and respectively \mathcal{B} and we focus on the image $H(A)$ of A , which is trivially a vector space because H is linear. On a side we consider the sub- C^* -algebra $\mathcal{B}_{H(A)}$ of \mathcal{B} generated by $H(A)$ (cfr. Remark 1.4.3). Since H is compatible with the multiplications and the involutions of \mathcal{A} and \mathcal{B} , it follows that $H(A)$ endowed with the restriction of the product and of the involution of \mathcal{B} is a $*$ -algebra with unit $H1_{\mathcal{A}} = 1_{\mathcal{B}}$ and the map $H' : A \rightarrow H(A)$, defined by $H'a = Ha$, is a $*$ -isomorphism from \mathcal{A} to the $*$ -algebra $H(A)$. We have seen that H is an injective isometry between the Banach spaces \mathcal{A} and \mathcal{B} . This allows us to recognize that $H(A)$ is a closed subspace of \mathcal{B} . Consider in fact a sequence $\{b_n\}$ of elements of the vector space $H(A)$ that converges to $b \in \mathcal{B}$ with respect to the norm of \mathcal{B} and take the sequence $\{a_n = H^{-1}b_n\}$ in \mathcal{A} : since $\{b_n\}$ is a Cauchy sequence in \mathcal{B} (as a consequence of being convergent) and H is an isometry, it follows that $\{a_n\}$ is a Cauchy sequence in \mathcal{A} :

$$\|a_n - a_m\| = \|Ha_n - Ha_m\| = \|b_n - b_m\|.$$

But \mathcal{A} is a Banach space and hence we find the limit $a \in \mathcal{A}$ of the sequence $\{a_n\}$ with respect to the norm of \mathcal{A} . Hence, bearing in mind that H is in particular

continuous between the Banach spaces \mathcal{A} and \mathcal{B} , we have the following situation:

$$Ha \xleftarrow{\infty \leftarrow n} Ha_n = b_n \xrightarrow{n \rightarrow \infty} b.$$

The uniqueness of the limit in \mathcal{B} implies that $Ha = b$, hence in particular $b \in H(A)$. This proves that $H(A)$ is actually a closed subspace of \mathcal{B} . Then the unital $*$ -algebra $H(A)$ endowed with the restriction of the norm of \mathcal{B} defines a unital sub-C*-algebra of \mathcal{B} (cfr. Definition 1.4.2) that we denote with $H(\mathcal{A})$. Since $\mathcal{B}_{H(A)}$ is by definition the smallest sub-C*-algebra of \mathcal{B} including $H(A)$ and the vector space underlying $H(\mathcal{A})$ coincides exactly with $H(A)$, we conclude that $H(\mathcal{A}) = \mathcal{B}_{H(A)}$. It also turns out that we have at our disposal two new morphisms of \mathbf{alg} :

- $H' : \mathcal{A} \rightarrow H(\mathcal{A})$, which is in particular a unit preserving $*$ -isomorphism between unital C*-algebras and hence, from Remark 1.4.8, an isometric isomorphism between the Banach spaces \mathcal{A} and $H(\mathcal{A})$ too;
- the inclusion map $\iota_{H(A)}^B$ of $H(A)$ in B , which is recognized to be an injective unit preserving $*$ -homomorphism between the unital C*-algebras $H(\mathcal{A})$ and \mathcal{B} : This is a consequence of a more general fact that holds for each C*-algebra \mathcal{C} and each sub-C*-algebra \mathcal{S} of \mathcal{C} , specifically that the inclusion map $\iota_{\mathcal{S}}^{\mathcal{C}}$ of the vector space S underlying \mathcal{S} in the vector space C underlying \mathcal{C} is recognized to be an injective unit preserving $*$ -homomorphism between the C*-algebras \mathcal{S} and \mathcal{C} .

Using the construction above, we can decompose each morphism $H : \mathcal{A} \rightarrow \mathcal{B}$ of \mathbf{alg} in the morphisms $\iota_{H(A)}^B \in \mathbf{Mor}_{\mathbf{alg}}(H(\mathcal{A}), \mathcal{B})$ and $H' \in \mathbf{Mor}_{\mathbf{alg}}(\mathcal{A}, H(\mathcal{A}))$ (which is also a $*$ -isomorphism) according to the formula $H = \iota_{H(A)}^B \circ H'$.

Now we are ready to check that \mathbf{ghs} and \mathbf{alg} are actually categories.

Remark 2.1.4. We begin from \mathbf{ghs} . If we take $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, $\mathcal{N} = (N, h, \mathfrak{p}, \mathfrak{u})$, $\mathcal{O} = (O, i, \mathfrak{q}, \mathfrak{v})$ in $\mathbf{Obj}_{\mathbf{ghs}}$ and $\phi \in \mathbf{Mor}_{\mathbf{ghs}}(\mathcal{M}, \mathcal{N})$, $\psi \in \mathbf{Mor}_{\mathbf{ghs}}(\mathcal{N}, \mathcal{O})$, we immediately realize that $\psi \circ \phi : M \rightarrow O$ is a smooth map and an immersion as a consequence of the same properties for $\psi \circ \phi$ and ψ . To prove that it is also an embedding with open image, in first place we must show that $(\psi \circ \phi)(M) = \psi(\phi(M))$ is an open subset of O . This is true because $\phi(M)$ is an open subset of N and ψ is an open map from N to O (see the end of Remark 1.1.7). After that one applies Remark 1.1.7 to $(\psi \circ \phi)(M)$, obtains a d -dimensional submanifold of O and realizes that $\psi \circ \phi$ is an embedding because $(\psi \circ \phi)'$ can be written as the composition of $\psi'|_{\phi(M)} : \phi(M) \rightarrow \psi(\phi(M))$ and $\phi' : M \rightarrow \phi(M)$, which are both diffeomorphisms. Then we must check $\psi \circ \phi$ is isometric and preserves orientation and time orientation. This can be directly checked exploiting the same properties that are assumed to hold for both ϕ and ψ . Now the question is whether the image of M through $\psi \circ \phi$ is a causally convex subset of \mathcal{O} or not. We try to give an answer fixing p ,

$q \in (\psi \circ \phi)(M)$. We take a causal curve γ in \mathcal{O} connecting p and q and we check that it is entirely contained in $(\psi \circ \phi)(M)$. Since p and q are obviously in $\psi(N)$, that is \mathcal{O} -causally convex by hypothesis, it follows that γ is contained in $\psi(N)$. Then we can use the isometric diffeomorphism ψ' to construct $\gamma' = \psi'^{-1} \circ \gamma$. This is an h -causal curve in N due to the fact that $\psi'^{-1} : \psi(\mathcal{N}) \rightarrow \mathcal{N}$ is an isometric diffeomorphism and it connects the points $p' = h'^{-1}(p)$ and $q' = h'^{-1}(q)$ of N . But p' and q' are also points of $\phi(M)$ since $p, q \in (\psi \circ \phi)(M)$. Then by the same argument applied to ϕ in place of ψ , we obtain that γ' is entirely contained in $\phi(M)$. From this we conclude that γ is contained in $(\psi \circ \phi)(M)$ and hence this subset of O is indeed \mathcal{O} -causally convex. This proves that $\psi \circ \phi$ is actually an element of $\text{Mor}_{\text{ghs}}(\mathcal{M}, \mathcal{O})$ and so the law of composition is well defined. We must still check that the category axioms hold. For each $\mathcal{M} \in \text{Obj}_{\text{ghs}}$ it is easy to check that the identity morphism is provided by the function $M \rightarrow M, p \mapsto p$ and so also the identity law is verified. As for the associativity of the composition law, it holds because the ordinary composition of functions is always associative.

Now we focus on **alg**. Here the situation is even simpler. Taking $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \text{Obj}_{\text{alg}}$ and $H \in \text{Mor}_{\text{alg}}(\mathcal{A}, \mathcal{B}), K \in \text{Mor}_{\text{alg}}(\mathcal{B}, \mathcal{C})$, we immediately realize that $K \circ H : \mathcal{A} \rightarrow \mathcal{C}$ makes sense and gives an injective unit preserving $*$ -homomorphism. In order to show the strategy of proof for the last statement, we explicitly prove that $H \circ K$ is actually compatible with the involutions of \mathcal{A} and of \mathcal{C} . Fix $a \in \mathcal{A}$. Since both H and K are $*$ -homomorphisms between the appropriate algebras by hypothesis, it follows that

$$(H \circ K)(a^*) = HK(a^*) = H((Ka)^*) = (HKa)^* = ((H \circ K)a)^*.$$

For each $\mathcal{A} \in \text{Obj}_{\text{alg}}$, we recognize the map $\mathcal{A} \rightarrow \mathcal{A}, a \mapsto a$ to be the identity morphism of \mathcal{A} . As before, the associativity of the composition law is trivial.

At this point we have at hand all the material needed to state the generally covariant locality principle.

2.1.2 Formulation of the generally covariant locality principle

The *generally covariant locality principle* (briefly *GCLP*) imposes that each quantum field theory on each globally hyperbolic spacetime must be formulated as a *locally covariant quantum field theory* (*LCQFT*).

Since we have not yet defined what it is meant for a LCQFT, the statement of the GCLP is still an empty box. We fill this box with the next definition and we take the chance to state two additional properties that can be required to a LCQFT. Later we will see that the fulfilment of these additional properties allows us to completely recover the Haag-Kastler axioms starting from the GCLP.

Definition 2.1.5. We call *locally covariant quantum field theory* (or *LCQFT*) any covariant functor \mathcal{A} from the category \mathbf{ghs} to the category \mathbf{alg} .

A locally covariant quantum field theory \mathcal{A} is said to be *causal* if the following condition (called *causality condition*) holds for each $\mathcal{M}_1 = (M_1, g_1, \mathfrak{o}_1, \mathfrak{t}_1)$, $\mathcal{M}_2 = (M_2, g_2, \mathfrak{o}_2, \mathfrak{t}_2)$, $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t}) \in \mathbf{Obj}_{\mathbf{ghs}}$, each $\psi_1 \in \mathbf{Mor}_{\mathbf{ghs}}(\mathcal{M}_1, \mathcal{M})$ and each $\psi_2 \in \mathbf{Mor}_{\mathbf{ghs}}(\mathcal{M}_2, \mathcal{M})$ such that $\psi_1(M_1)$ and $\psi_2(M_2)$ are \mathcal{M} -causally separated subsets of M :

the elements of the image through the morphism $\mathcal{A}(\psi_1)$ of the object $\mathcal{A}(\mathcal{M}_1)$ commute with the elements of the image through the morphism $\mathcal{A}(\psi_2)$ of the object $\mathcal{A}(\mathcal{M}_2)$, i.e.

$$[\mathcal{A}(\psi_1)(\mathcal{A}(\mathcal{M}_1)), \mathcal{A}(\psi_2)(\mathcal{A}(\mathcal{M}_2))] = \{0\},$$

where 0 is the zero element of the C^* -algebra $\mathcal{A}(\mathcal{M})$.

Moreover \mathcal{A} is said to fulfil the *time slice axiom* if the following condition holds for each $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, $\mathcal{N} \in \mathbf{ghs}$ and each $\psi \in \mathbf{Mor}_{\mathbf{ghs}}(\mathcal{M}, \mathcal{N})$ such that $\psi(M)$ contains a smooth spacelike Cauchy surface for \mathcal{N} :

the morphism $\mathcal{A}(\psi)$ is surjective, i.e.

$$\mathcal{A}(\psi)(\mathcal{A}(\mathcal{M})) = \mathcal{A}(\mathcal{N}).$$

Remark 2.1.6. Even if a precise discussion on the physical meaning of the generally covariant locality principle could be conducted after the recovering of the algebraic quantum field theory framework proposed by Haag and Kastler (cfr. [HK64]) simply borrowing the interpretation of the Haag-Kastler axioms, we want to make some considerations on the last definition (as a matter of fact on the GCLP) from now.

The first thing that we notice is that the functorial structure of any locally covariant quantum field theory implements a sort of geometrical locality in quantum field theory. We realize this fact considering a LCQFT $\mathcal{A} : \mathbf{ghs} \rightarrow \mathbf{alg}$, an arbitrary globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ and a \mathcal{M} -causally convex connected open subset Ω of M . From the last part of Remark 2.1.2 we deduce that $\iota_\Omega^M \in \mathbf{Mor}_{\mathbf{ghs}}(\mathcal{M}|_\Omega, \mathcal{M})$, hence we consider $\mathcal{A}(\iota_\Omega^M)$, which is a morphism of \mathbf{alg} from $\mathcal{A}(\mathcal{M}|_\Omega)$ to $\mathcal{A}(\mathcal{M})$, and we focus on its image $\mathcal{A}(\iota_\Omega^M)(\mathcal{A}(\mathcal{M}|_\Omega))$. Recalling Remark 2.1.3, we realize that $\mathcal{A}(\iota_\Omega^M)(\mathcal{A}(\mathcal{M}|_\Omega))$ is a unital sub- C^* -algebra of the unital C^* -algebra $\mathcal{A}(\mathcal{M})$. This is exactly what we mean by geometrical locality: A causally convex connected open subset of a globally hyperbolic spacetime, when intended as a globally hyperbolic spacetime in its own right, is associated by a LCQFT \mathcal{A} to a unital C^* -algebra whose image (through the morphism of \mathbf{ghs} obtained via \mathcal{A} from the inclusion map of Ω in M) is a unital sub- C^* -algebra of the unital C^* -algebra associated via \mathcal{A} to the entire globally hyperbolic spacetime.

This geometrical locality allows us to introduce a physical interpretation. We assume that, given a LCQFT \mathcal{A} and a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, for each \mathcal{M} -causally convex relatively compact connected open subset Ω of M , the unital sub-C*-algebra $\mathcal{A}(\iota_\Omega^M) \mathcal{A}(\mathcal{M}|_\Omega)$ of the full unital C*-algebra $\mathcal{A}(\mathcal{M})$ is the mathematical representation of the quantum observables that could be measured on Ω . Notice that this interpretation cannot be applied to the full algebra $\mathcal{A}(\mathcal{M})$ because M cannot be compact (if it were, it would violate the causality condition, cfr. [O’N83, Chap. 14, Lem. 13, p. 407]). By this assumption we mean that we consider as physical observables only those that can be measured on “small” regions of the spacetime (precisely \mathcal{M} -causally convex relatively compact connected open subsets of M). Such choice is done because it doesn’t appear physically sensible to deal with an observable on a too large region since we are not able to realize an experimental apparatus that makes measurements for an observable “everywhere in space and time”, or anyway on a region too much extended “in space” or “in time” (or both). The entire algebra of quantum observables on a given globally hyperbolic spacetime is obtained as the unital sub-C*-algebra of $\mathcal{A}(\mathcal{M})$ generated by all the observables that we classified as physical. We use this interpretation to explore the physical meaning of some properties of a locally covariant quantum field theory.

Returning to the definition of a LCQFT, we notice that it is nothing but a covariant functor from \mathbf{ghs} to \mathbf{alg} , which is to say that the GCLP simply states that each quantum field theory must be formulated as a covariant functor that assigns a unital C*-algebra to each globally hyperbolic spacetime and an injective unit preserving *-homomorphism between the appropriate unital C*-algebras to each orientation and time orientation preserving isometric embedding between globally hyperbolic spacetimes whose image is a causally convex open subset of the target spacetime. The physical sense that we obtain in light of our interpretation is the following: For each globally hyperbolic spacetime and each “sufficiently small” region, we have a unital sub-C*-algebra that represents the quantum observables on that region and all these unital sub-C*-algebras generate the entire algebra of observables on the given globally hyperbolic spacetime. The power of the GCLP resides in this fact, that is the possibility of discussing a quantum field theory on all the globally hyperbolic spacetimes at once.

This functorial structure automatically incorporates in quantum field theory the notion of general covariance under the transformations induced by a group of isometric diffeomorphisms of the globally hyperbolic spacetime. We will see this in detail when the Haag-Kastler axioms will be recovered. In our interpretation this means that we expect to find a representation of the group of isometric diffeomorphisms in terms of a group of automorphisms on the algebra of observables and that we require that such representation satisfies covariance (as intended in the language of category theory).

To give a physical interpretation of the property of geometrical locality encountered before, we proceed in the following way. Let Ω and Θ be \mathcal{M} -causally convex relatively compact connected open subsets of M such that $\Omega \subseteq \Theta$. We can consider the globally hyperbolic spacetime $\mathcal{M}|_{\Theta} = (\Theta, g|_{\Theta}, \mathfrak{o}|_{\Theta}, \mathfrak{t}|_{\Theta})$ and we immediately recognize that Ω is a $\mathcal{M}|_{\Theta}$ -causally convex connected open subsets of Θ , so that we can also consider the globally hyperbolic spacetime $\mathcal{M}|_{\Theta}|_{\Omega}$, which coincides with $\mathcal{M}|_{\Omega}$ as it is easily seen. Hence we can consider the inclusion map ι_{Ω}^{Θ} and we realize that this is a morphism of **ghs** from $\mathcal{M}|_{\Theta}|_{\Omega} = \mathcal{M}|_{\Omega}$ to $\mathcal{M}|_{\Theta}$. This leads us to the conclusion that $\mathcal{A}(\iota_{\Omega}^{\Theta})(\mathcal{A}(\mathcal{M}|_{\Omega}))$ is a unital sub-C*-algebra of the unital C*-algebra $\mathcal{A}(\mathcal{M}|_{\Theta})$. This suggests that a sort of isotony holds for the algebras of observables associated to proper regions of a globally hyperbolic spacetime: If Ω is smaller than Θ , then we expect that the algebra of observables on Ω is a subalgebra of the algebra of observables of Θ (and both are trivially subalgebras of the complete algebra of observables associated to the given globally hyperbolic spacetime).

Now we turn our attention to the causality condition. We begin observing that the causality condition makes sense because of the functorial structure of each LCQFT \mathcal{A} : Taking three objects \mathcal{M} , $\mathcal{M}_1 = (M_1, g_1, \mathfrak{o}_1, \mathfrak{t}_1)$ and $\mathcal{M}_2 = (M_2, g_2, \mathfrak{o}_2, \mathfrak{t}_2)$ and two morphisms $\psi_1 : \mathcal{M}_1 \rightarrow \mathcal{M}$ and $\psi_2 : \mathcal{M}_2 \rightarrow \mathcal{M}$ such that $\psi_1(M_1)$ and $\psi_2(M_2)$ are \mathcal{M} -causally separated, we can evaluate the commutator of an element of $\mathcal{A}(\psi_1)(\mathcal{A}(\mathcal{M}_1))$ with an element of $\mathcal{A}(\psi_2)(\mathcal{A}(\mathcal{M}_2))$ because, owing to the functorial structure, both $\mathcal{A}(\psi_1)(\mathcal{A}(\mathcal{M}_1))$, $\mathcal{A}(\psi_2)(\mathcal{A}(\mathcal{M}_2))$ are unital sub-C*-algebras of $\mathcal{A}(\mathcal{M})$.

From a physical point of view the causality condition imposes some restrictions to the causal structure of a LCQFT \mathcal{A} . We can sketch the typology of such restrictions considering the globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ and two \mathcal{M} -causally convex relatively compact connected open subsets Ω and Θ of M that are \mathcal{M} -causally separated. As usual we interpret Ω and Θ as been globally hyperbolic spacetimes in their own right (denoted respectively by $\mathcal{M}|_{\Omega}$ and $\mathcal{M}|_{\Theta}$) and we take into account the inclusion maps ι_{Ω}^M and ι_{Θ}^M (which are actually morphisms of **ghs** respectively from $\mathcal{M}|_{\Omega}$ and from $\mathcal{M}|_{\Theta}$ to \mathcal{M}). The causality condition imposes that

$$[\mathcal{A}(\iota_{\Omega}^M)(\mathcal{A}(\mathcal{M}|_{\Omega})), \mathcal{A}(\iota_{\Theta}^M)(\mathcal{A}(\mathcal{M}|_{\Theta}))] = \{0\}.$$

In light of our interpretation of the unital sub-C*-algebras associated to proper regions as the algebras of the quantum observables on these regions, the last equation means that the observables associated to (causally convex relatively compact connected open) subsets which are causally separated should be measurable independently. From physical considerations this property is expected to hold for each quantum field theory: we hardly admit a physical theory in which there are observables associated to causally separated regions that cannot be measured independently. Hence we may see the causality condition as a restriction on the possible

correlations between observables localized in proper domains which are causally separated.

The time slice axiom seems to be a condition on the causal structure of a LCQFT too. Consider a LCQFT \mathcal{A} and a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$. From Theorem 1.2.15 we deduce that there exists a smooth spacelike Cauchy surface Σ for \mathcal{M} . If we choose a causally convex connected open subset Ω of M including Σ , taking into account the globally hyperbolic spacetime $\mathcal{M}|_\Omega$ and the morphism $\iota_\Omega^M : \mathcal{M}|_\Omega \rightarrow \mathcal{M}$ of \mathfrak{ghs} , we see that the time slice axiom imposes that

$$\mathcal{A}(\iota_\Omega^M)(\mathcal{A}(\mathcal{M})) = \mathcal{A}(\mathcal{M}).$$

To give an interpretation of the time slice axiom in terms of quantum observables, we must consider a \mathcal{M} -causally convex relatively compact connected open subset Θ of M and we think to it as being itself a globally hyperbolic spacetime denoted by $\mathcal{M}|_\Theta$. Applying Remark 1.2.17 to $\mathcal{M}|_\Theta$, we obtain for $\varepsilon > 0$ an $\mathcal{M}|_\Theta$ -causally convex connected open subset Ω_ε of Θ that includes a Cauchy surface of $\mathcal{M}|_\Theta$. The closure of Ω_ε in M is included in the closure of Θ in M , which is compact in M by hypothesis. Therefore Ω_ε is relatively compact in M . This proves the existence of $\mathcal{M}|_\Theta$ -causally convex relatively compact connected open subsets of Θ that include Cauchy surfaces of $\mathcal{M}|_\Theta$. We choose a subset with these properties and we denote it with Ω . We recognize that Ω is also \mathcal{M} -causally convex and that the globally hyperbolic spacetimes $\mathcal{M}|_\Omega$ and $\mathcal{M}|_\Theta|_\Omega$ coincide so that we can consider the inclusion map ι_Ω^Θ as a morphism of \mathfrak{ghs} from $\mathcal{M}|_\Omega$ to $\mathcal{M}|_\Theta$. In the present situation the time slice axiom imposes that

$$\mathcal{A}(\iota_\Omega^\Theta)(\mathcal{A}(\mathcal{M}|_\Omega)) = \mathcal{A}(\mathcal{M}|_\Theta).$$

This relation means that, when Θ is a proper subset of some globally hyperbolic spacetime \mathcal{M} and Ω is a proper subset of Θ including a Cauchy surface of $\mathcal{M}|_\Theta$, the quantum observables over Ω exhaust all the quantum observables that are admitted by the physics on Θ , even if Θ is larger. Then the time slice axiom forces the physics over a proper subset Θ of a globally hyperbolic spacetime to be completely determined by the physics over a proper neighborhood Ω of a Cauchy surface for $\mathcal{M}|_\Theta$.

The functorial approach of the GCLP allows us to introduce a notion of equivalence between LCQFTs.

Definition 2.1.7. Let \mathcal{A} and \mathcal{B} be two LCQFTs. We say that \mathcal{A} and \mathcal{B} are equivalent if there exists a covariant natural isomorphism $\mathfrak{i} : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$.

The reader can easily check that this is an equivalence relation on the set of LCQFTs. Such equivalence can be interpreted as physical indistinguishability. Suppose that \mathcal{A} and \mathcal{B} are LCQFTs and that \mathfrak{i} is covariant natural isomorphism

from \mathcal{A} to \mathcal{B} and fix two globally hyperbolic spacetimes \mathcal{M}, \mathcal{N} and a morphism $\psi : \mathcal{M} \rightarrow \mathcal{N}$ of **ghs**. We have that $\mathcal{A}(\mathcal{M})$ and $\mathcal{B}(\mathcal{M})$ may be identified through the unit preserving *-isomorphism $\mathbf{i}_{\mathcal{M}} : \mathcal{A}(\mathcal{M}) \rightarrow \mathcal{B}(\mathcal{M})$ (similarly we can identify $\mathcal{A}(\mathcal{N})$ and $\mathcal{B}(\mathcal{N})$ through the unit preserving *-isomorphism $\mathbf{i}_{\mathcal{N}} : \mathcal{A}(\mathcal{N}) \rightarrow \mathcal{B}(\mathcal{N})$) and that the injective unit preserving *-homomorphisms $\mathcal{A}(\psi)$ and $\mathcal{B}(\psi)$ satisfy the following relation:

$$\mathbf{i}_{\mathcal{N}} \circ \mathcal{A}(\psi) = \mathcal{B}(\psi) \circ \mathbf{i}_{\mathcal{M}}.$$

Then, with the above identifications, also $\mathcal{A}(\psi)$ and $\mathcal{B}(\psi)$ are identified. This identification in our interpretation means that the quantum observables admitted by the physics described by the theory \mathcal{A} on some globally hyperbolic spacetime are exactly the same as those admitted by the physics described by the theory \mathcal{B} on the same globally hyperbolic spacetime, that is to say that the physics described by \mathcal{A} is exactly the same as the physics described by \mathcal{B} on each globally hyperbolic spacetime.

2.1.3 Recovering the Haag-Kastler framework

In this subsection we check that our approach to quantum field theory through the generally covariant locality principle leads us to the complete recovery of the *Haag-Kastler axioms* for each globally hyperbolic spacetime. By this we mean that each locally covariant quantum field theory applied to an arbitrary globally hyperbolic spacetime gives rise to a quantum field theory for that spacetime in the formulation suggested by Haag-Kastler in their seminal paper [HK64]. We underline that, this formulation of quantum field theory, known as *algebraic quantum field theory*, although being equivalent to the traditional formulation, has the advantage of being stated in a rigorous mathematical framework, specifically that of C*-algebras.

A relevant part of the problem of recovering the algebraic approach to quantum field theory has already been discussed in Remark 2.1.6 even if we did not stress this fact there. In the next theorem we will complete this discussion so that it will become evident by comparison with [HK64] that the Haag-Kastler axioms are recovered on each globally hyperbolic spacetime once that a LCQFT is given.

We begin with a definition.

Definition 2.1.8. Let \mathcal{A} be a LCQFT and let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be a globally hyperbolic spacetime. We define the set $\mathcal{K}_{\mathcal{M}}$ of all \mathcal{M} -causally convex non empty relatively compact connected open subsets of M and the family $\{\mathcal{A}_{\mathcal{M}}(\Omega)\}$ consisting of the unital sub-C*-algebras $\mathcal{A}_{\mathcal{M}}(\Omega) = \mathcal{A}(\iota_{\Omega}^M(\mathcal{A}(\mathcal{M}|_{\Omega})))$, called *local algebras*, of the unital C*-algebra $\mathcal{A}(\mathcal{M})$ for $\Omega \in \mathcal{K}_{\mathcal{M}}$. Moreover we define $\mathcal{A}_{\mathcal{M}}$ as the unital sub-C*-algebra of $\mathcal{A}(\mathcal{M})$ generated by the family $\{\mathcal{A}_{\mathcal{M}}(\Omega)\}$.

Notice that the elements of $\mathcal{K}(\mathcal{M})$ are exactly those subsets of \mathcal{M} that we used

in our interpretation of the GCLP (cfr. Remark 2.1.6) to pick out the physically acceptable observables on the globally hyperbolic spacetime \mathcal{M} . There we did not specified the exclusion of the empty set, however it appears obvious from a physical point of view that it does not make sense to speak of the physics on a region with no events.

In that context we already noticed that, for each $\Omega \in \mathcal{K}_{\mathcal{M}}$, $\mathcal{M}|_{\Omega}$ is actually a globally hyperbolic spacetime, so that we can consider the unital C*-algebra $\mathcal{A}(\mathcal{M}|_{\Omega})$ and the morphism ι_{Ω}^M of the category \mathbf{ghs} . Then we can actually define $\mathcal{A}_{\mathcal{M}}(\Omega)$ as above and we recognize that it is a unital sub-C*-algebra of the larger unital C*-algebra $\mathcal{A}(\mathcal{M})$. This shows that the family $\{\mathcal{A}_{\mathcal{M}}(\Omega)\}$ is well defined. In our interpretation we also specified that we cannot consider $\mathcal{A}(\mathcal{M})$ as an algebra of observables because M cannot be compact otherwise \mathcal{M} would violate the causality condition (cfr. [O’N83, Chap. 14, Lem. 13, p. 407]). For the same reason $\mathcal{A}(\mathcal{M})$ is not included in the family $\{\mathcal{A}_{\mathcal{M}}(\Omega)\}$.

When we define $\mathcal{A}_{\mathcal{M}}$ as the sub-C*-algebra of $\mathcal{A}(\mathcal{M})$ generated by the family $\{\mathcal{A}_{\mathcal{M}}(\Omega)\}$, we intend that $\mathcal{A}_{\mathcal{M}}$ is the sub-C*-algebra of $\mathcal{A}(\mathcal{M})$ generated by the subset

$$S = \bigcup_{\Omega \in \mathcal{K}_{\mathcal{M}}} \mathcal{A}_{\mathcal{M}}(\Omega)$$

of $\mathcal{A}(\mathcal{M})$ (refer to 1.4.3 for the notion of generated sub-C*-algebra). That this definition actually makes sense is assured by the fact that all elements of $\{\mathcal{A}_{\mathcal{M}}(\Omega)\}$ are sub-C*-algebras of $\mathcal{A}(\mathcal{M})$.

With the last definition we are ready to formulate the theorem that recovers the Haag-Kastler axioms starting from a LCQFT applied to an arbitrary globally hyperbolic spacetime.

Theorem 2.1.9. *Let \mathcal{A} be a LCQFT and let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be a globally hyperbolic spacetime. Consider $\mathcal{K}_{\mathcal{M}}$, $\{\mathcal{A}_{\mathcal{M}}(\Omega)\}$ and $\mathcal{A}_{\mathcal{M}}$ as defined above. Then the Haag-Kastler axioms (cfr. [HK64]) are fully recovered. Specifically the following properties hold:*

- *isotony: for each $\Omega, \Theta \in \mathcal{K}_{\mathcal{M}}$ such that $\Omega \subseteq \Theta$, $\mathcal{A}_{\mathcal{M}}(\Omega)$ is a sub-C*-algebra of $\mathcal{A}_{\mathcal{M}}(\Theta)$;*
- *common unit: all the elements of $\{\mathcal{A}_{\mathcal{M}}(\Omega)\}$ have a common unit;*
- *algebra of observables: $\mathcal{A}_{\mathcal{M}}$ is the closure in $\mathcal{A}(\mathcal{M})$ of the union of the family $\{\mathcal{A}_{\mathcal{M}}(\Omega)\}$;*
- *covariance: if G is a group of orientation and time orientation preserving isometric diffeomorphisms of \mathcal{M} , then there exists a representation of G in terms of *-automorphisms on $\mathcal{A}_{\mathcal{M}}$ such that, for each $f \in G$ and each $\Omega \in \mathcal{K}_{\mathcal{M}}$,*

the $*$ -automorphism α_f associated to f satisfies the condition

$$\alpha_f(\mathcal{A}_{\mathcal{M}}(\Omega)) = \mathcal{A}_{\mathcal{M}}(f(\Omega));$$

- *local commutativity*: if \mathcal{A} is causal then, for each $\Omega, \Theta \in \mathcal{K}_{\mathcal{M}}$ such that Ω and Θ are \mathcal{M} -causally separated, we have that

$$[\mathcal{A}_{\mathcal{M}}(\Omega), \mathcal{A}_{\mathcal{M}}(\Theta)] = \{0\};$$

- *time slice axiom*: if \mathcal{A} fulfils the time slice axiom, Σ is a smooth spacelike Cauchy surface for \mathcal{M} and S is a connected open subset of Σ such that its Cauchy development $D_{\mathcal{M}}(S)$ is relatively compact, then for each $\Omega \in \mathcal{K}_{\mathcal{M}}$ such that $S \subseteq \Omega$ we have

$$\mathcal{A}_{\mathcal{M}}(\Omega) \supseteq \mathcal{A}_{\mathcal{M}}(D_{\mathcal{M}}(S)).$$

Proof. We start from isotony. Suppose that Ω and Θ are elements of $\mathcal{K}_{\mathcal{M}}$ such that $\Omega \subseteq \Theta$. In Remark 2.1.6 we showed that $\mathcal{A}(\iota_{\Omega}^{\Theta})(\mathcal{A}(\mathcal{M}|_{\Omega}))$ is a unital sub-C*-algebra of the unital C*-algebra $\mathcal{A}(\mathcal{M}|_{\Theta})$. If we consider the morphisms ι_{Ω}^M and ι_{Θ}^M of the category **ghs**, we immediately recognize that $\iota_{\Omega}^M = \iota_{\Theta}^M \circ \iota_{\Omega}^{\Theta}$. Since \mathcal{A} is a covariant functor, we have that $\mathcal{A}(\iota_{\Omega}^M) = \mathcal{A}(\iota_{\Theta}^M) \circ \mathcal{A}(\iota_{\Omega}^{\Theta})$. We deduce that

$$\begin{aligned} \mathcal{A}_{\mathcal{M}}(\Omega) &= \mathcal{A}(\iota_{\Omega}^M)(\mathcal{A}(\mathcal{M}|_{\Omega})) \\ &= (\mathcal{A}(\iota_{\Theta}^M) \circ \mathcal{A}(\iota_{\Omega}^{\Theta}))(\mathcal{A}(\mathcal{M}|_{\Omega})) \\ &\subseteq \mathcal{A}(\iota_{\Theta}^M)(\mathcal{A}(\mathcal{M}|_{\Theta})) \\ &= \mathcal{A}_{\mathcal{M}}(\Theta). \end{aligned}$$

Since both $\mathcal{A}_{\mathcal{M}}(\Omega)$ and $\mathcal{A}_{\mathcal{M}}(\Theta)$ are unital sub-C*-algebras of $\mathcal{A}(\mathcal{M})$, the inclusion $\mathcal{A}_{\mathcal{M}}(\Omega) \subseteq \mathcal{A}_{\mathcal{M}}(\Theta)$ implies that $\mathcal{A}_{\mathcal{M}}(\Omega)$ is a unital sub-C*-algebra of $\mathcal{A}_{\mathcal{M}}(\Theta)$.

Now we turn our attention to the units of the elements of the family $\{\mathcal{A}_{\mathcal{M}}(\Omega)\}$. Let Ω and Θ be two arbitrary elements of $\mathcal{K}_{\mathcal{M}}$. Applying Remark 2.1.2, we can consider the globally hyperbolic spacetimes $\mathcal{M}|_{\Omega}$ and $\mathcal{M}|_{\Theta}$ and the morphisms ι_{Ω}^M and ι_{Θ}^M of **ghs**. Using \mathcal{A} , we obtain the corresponding morphisms $\mathcal{A}(\iota_{\Omega}^M)$ and $\mathcal{A}(\iota_{\Theta}^M)$ of **alg** that map each element of the unital C*-algebra $\mathcal{A}(\mathcal{M}|_{\Omega})$ and respectively $\mathcal{A}(\mathcal{M}|_{\Theta})$ into an element of the unital C*-algebra $\mathcal{A}(\mathcal{M})$. Denoting with 1_{Ω} the unit of $\mathcal{A}(\mathcal{M}|_{\Omega})$, with 1_{Θ} the unit of $\mathcal{A}(\mathcal{M}|_{\Theta})$ and with 1_M the unit of $\mathcal{A}(\mathcal{M})$ and keeping in mind that all morphisms of **alg** are unit preserving, i.e. they map the unit of their domain algebra to the unit of their codomain algebra, we conclude that

$$\mathcal{A}(\iota_{\Omega}^M)1_{\Omega} = 1_M = \mathcal{A}(\iota_{\Theta}^M)1_{\Theta}.$$

From Remark 2.1.3 we notice that $\mathcal{A}(\iota_\Omega^M)1_\Omega$ and $\mathcal{A}(\iota_\Theta^M)1_\Theta$ are respectively the units of $\mathcal{A}_\mathcal{M}(\Omega)$ and $\mathcal{A}_\mathcal{M}(\Theta)$, so that the last equation means that the unit of $\mathcal{A}_\mathcal{M}(\Omega)$ coincides with the unit of $\mathcal{A}_\mathcal{M}(\Theta)$.

$\mathcal{A}_\mathcal{M}$ is defined as the sub-C*-algebra of $\mathcal{A}(\mathcal{M})$ that is generated by the set

$$S = \bigcup_{\Omega \in \mathcal{K}_\mathcal{M}} \mathcal{A}_\mathcal{M}(\Omega).$$

Consider a and b in S . Then a is in $\mathcal{A}_\mathcal{M}(\Omega)$ for some $\Omega \in \mathcal{K}_\mathcal{M}$ and b is in $\mathcal{A}_\mathcal{M}(\Theta)$ for some $\Theta \in \mathcal{K}_\mathcal{M}$. Since both Ω and Θ are relatively compact, we have that $K = \overline{\Omega} \cup \overline{\Theta}$ is compact and so we can apply the fourth point of Proposition 1.2.18 to K so that we find $\Delta \in \mathcal{K}_\mathcal{M}$ including K . In particular both Ω and Θ are included in Δ and hence isotony implies that a and b are also elements of $\mathcal{A}_\mathcal{M}(\Delta)$. Then we can take linear combinations, products and involutions with them and we will always get elements of $\mathcal{A}_\mathcal{M}(\Delta)$ since it is a C*-algebra. But $\mathcal{A}_\mathcal{M}(\Delta)$ is included in S too, so linear combination, product and involution are internal operations on S . Therefore S is a vector space endowed with two internal operations that are our candidates for being a multiplication and an involution. They are actually such because they fulfil the properties that qualify them as a multiplication and an involution on the larger vector space $\mathcal{A}(\mathcal{M})$. Hence we can think of S as a unital *-algebra (its unit being 1_M as a consequence of what we have seen above). When we endow S with the norm of $\mathcal{A}(\mathcal{M})$, we realize that it lacks only of closure in $\mathcal{A}(\mathcal{M})$ to become a unital C*-algebra itself. So we close S in $\mathcal{A}(\mathcal{M})$ and we denote with $\mathcal{A}'_\mathcal{M}$ the unital C*-algebra that we obtain. By construction $S \subseteq \mathcal{A}'_\mathcal{M}$, hence $\mathcal{A}_\mathcal{M} \subseteq \mathcal{A}'_\mathcal{M}$ by definition of $\mathcal{A}_\mathcal{M}$ as the sub-C*-algebra of $\mathcal{A}(\mathcal{M})$ generated by S . We want to prove that $\mathcal{A}_\mathcal{M} \supseteq \mathcal{A}'_\mathcal{M}$. To this end pick $a \in \mathcal{A}'_\mathcal{M}$. By construction a is the limit in the norm of $\mathcal{A}(\mathcal{M})$ of a sequence $\{a_n\}$ of elements of S that is Cauchy with respect to the norm of $\mathcal{A}(\mathcal{M})$. Yet $S \subseteq \mathcal{A}_\mathcal{M}$ and the norm of $\mathcal{A}_\mathcal{M}$ is exactly the restriction of the norm of $\mathcal{A}(\mathcal{M})$ because $\mathcal{A}_\mathcal{M}$ is a sub-C*-algebra of $\mathcal{A}(\mathcal{M})$. We deduce that $\{a_n\}$ is also a Cauchy sequence in $\mathcal{A}_\mathcal{M}$. But, being a C*-algebra, $\mathcal{A}_\mathcal{M}$ is also a Banach space and so we find a limit b . Then $\{a_n\}$ converges to both a and b in $\mathcal{A}(\mathcal{M})$ and hence $a = b$. We conclude that $a \in \mathcal{A}_\mathcal{M}$, therefore $\mathcal{A}'_\mathcal{M} \subseteq \mathcal{A}_\mathcal{M}$.

As for covariance, we proceed in the following way. First of all we notice that the group G consists of bijective morphisms of **ghs** from \mathcal{M} to \mathcal{M} whose inverses are morphisms too: In order to recognize that $f \in G$ is a morphism of **ghs** we must only check that its image is \mathcal{M} -causally convex, but this is trivial because $f(M) = M$; bijectivity of $f \in G$ is assumed by hypothesis and its inverse f^{-1} is automatically a morphism of **ghs**. At this point we can use the LCQFT \mathcal{A} to map each $f \in G$ to a morphism of **alg**. From f^{-1} we obtain its inverse morphism so that $\mathcal{A}(f)$ is a

bijjective morphism of \mathbf{alg} from $\mathcal{A}(\mathcal{M})$ to $\mathcal{A}(\mathcal{M})$ whose inverse is a morphism too:

$$\begin{aligned}\mathcal{A}(f) \circ \mathcal{A}(f^{-1}) &= \mathcal{A}(f \circ f^{-1}) = \mathcal{A}(\text{id}_{\mathcal{M}}) = \text{id}_{\mathcal{A}(\mathcal{M})}; \\ \mathcal{A}(f^{-1}) \circ \mathcal{A}(f) &= \mathcal{A}(f^{-1} \circ f) = \mathcal{A}(\text{id}_{\mathcal{M}}) = \text{id}_{\mathcal{A}(\mathcal{M})}.\end{aligned}$$

Fix $f \in G$ and $\Omega \in \mathcal{K}_{\mathcal{M}}$. Observe that $f(\Omega) \in \mathcal{K}_{\mathcal{M}}$: It is a relatively compact open subset of M because it is the preimage of the relatively compact open subset Ω of M through the continuous map f^{-1} , it is connected because f is continuous and Ω is connected and finally it is \mathcal{M} -causally convex because f^{-1} is smooth and isometric and Ω is \mathcal{M} -causally convex. As usual, we can consider the globally hyperbolic spacetimes $\mathcal{M}|_{\Omega} = (\Omega, g|_{\Omega}, \mathfrak{o}|_{\Omega}, \mathfrak{t}|_{\Omega})$ and $\mathcal{M}|_{f(\Omega)} = (\Omega, g|_{f(\Omega)}, \mathfrak{o}|_{f(\Omega)}, \mathfrak{t}|_{f(\Omega)})$ and the morphisms ι_{Ω}^M and $\iota_{f(\Omega)}^M$ of \mathbf{ghs} . If we define the map $f_{\Omega} : \Omega \rightarrow f(\Omega)$, $p \mapsto f(p)$, as a consequence of the properties of f , we recognize that f_{Ω} is an orientation and time orientation preserving isometric diffeomorphism from $\mathcal{M}|_{\Omega}$ to $\mathcal{M}|_{f(\Omega)}$:

$$\begin{aligned}f_{\Omega} &\in \text{Mor}_{\mathbf{ghs}}(\mathcal{M}|_{\Omega}, \mathcal{M}|_{f(\Omega)}), \\ f_{\Omega}^{-1} &\in \text{Mor}_{\mathbf{ghs}}(\mathcal{M}|_{f(\Omega)}, \mathcal{M}|_{\Omega}).\end{aligned}$$

Then it follows that

$$\begin{aligned}\mathcal{A}(f_{\Omega}) &\in \text{Mor}_{\mathbf{alg}}(\mathcal{A}(\mathcal{M}|_{\Omega}), \mathcal{A}(\mathcal{M}|_{f(\Omega)})), \\ \mathcal{A}(f_{\Omega}^{-1}) &\in \text{Mor}_{\mathbf{alg}}(\mathcal{A}(\mathcal{M}|_{f(\Omega)}), \mathcal{A}(\mathcal{M}|_{\Omega}))\end{aligned}$$

are inverses one of the other. In particular we have that $\mathcal{A}(f_{\Omega})$ is surjective:

$$\mathcal{A}(f_{\Omega})(\mathcal{A}(\mathcal{M}|_{\Omega})) = \mathcal{A}(\mathcal{M}|_{f(\Omega)}).$$

It is easy to check that $\iota_{f(\Omega)}^M \circ f_{\Omega} = f \circ \iota_{\Omega}^M$ and hence we have

$$\mathcal{A}(\iota_{f(\Omega)}^M) \circ \mathcal{A}(f_{\Omega}) = \mathcal{A}(f) \circ \mathcal{A}(\iota_{\Omega}^M).$$

Therefore we find

$$\begin{aligned}\mathcal{A}_{\mathcal{M}}(f(\Omega)) &= \mathcal{A}(\iota_{f(\Omega)}^M)(\mathcal{A}(\mathcal{M}|_{f(\Omega)})) \\ &= (\mathcal{A}(\iota_{f(\Omega)}^M) \circ \mathcal{A}(f_{\Omega}))(\mathcal{A}(\mathcal{M}|_{\Omega})) \\ &= (\mathcal{A}(f) \circ \mathcal{A}(\iota_{\Omega}^M))(\mathcal{A}(\mathcal{M}|_{\Omega})) \\ &= \mathcal{A}(f)(\mathcal{A}_{\mathcal{M}}(\Omega)).\end{aligned}$$

Above we observed that $f(\Omega) \in \mathcal{K}_{\mathcal{M}}$ for each $\Omega \in \mathcal{K}_{\mathcal{M}}$. A similar argument applied to f^{-1} tells us also that $f^{-1}(\Omega) \in \mathcal{K}_{\mathcal{M}}$ for each $\Omega \in \mathcal{K}_{\mathcal{M}}$. This observation, together

with the last formula, implies that

$$\bigcup_{\Omega \in \mathcal{K}_{\mathcal{M}}} \mathcal{A}(f)(\mathcal{A}_{\mathcal{M}}(\Omega)) = \bigcup_{\Omega \in \mathcal{K}_{\mathcal{M}}} \mathcal{A}_{\mathcal{M}}(f(\Omega)) = \bigcup_{\Omega' \in \mathcal{K}_{\mathcal{M}}} \mathcal{A}_{\mathcal{M}}(\Omega').$$

Applying the third point of this theorem and bearing in mind that $\mathcal{A}(f)$ is continuous with respect to the norm of $\mathcal{A}(\mathcal{M})$, we draw the following conclusion:

$$\begin{aligned} \mathcal{A}(f)(\mathcal{A}_{\mathcal{M}}) &= \mathcal{A}(f)\left(\overline{\bigcup_{\Omega \in \mathcal{K}_{\mathcal{M}}} \mathcal{A}_{\mathcal{M}}(\Omega)}\right) = \overline{\bigcup_{\Omega \in \mathcal{K}_{\mathcal{M}}} \mathcal{A}(f)(\mathcal{A}_{\mathcal{M}}(\Omega))} = \overline{\bigcup_{\Omega' \in \mathcal{K}_{\mathcal{M}}} \mathcal{A}_{\mathcal{M}}(\Omega')} \\ &= \mathcal{A}_{\mathcal{M}}. \end{aligned}$$

The last equation implies that for each $f \in G$ we can define the map

$$\begin{aligned} \alpha_f : \mathcal{A}_{\mathcal{M}} &\rightarrow \mathcal{A}_{\mathcal{M}} \\ a &\mapsto \mathcal{A}(f)a \end{aligned}$$

and realize that it is a *-automorphism on the unital C*-algebra $\mathcal{A}_{\mathcal{M}}$ satisfying $\alpha_f(\mathcal{A}_{\mathcal{M}}) = \mathcal{A}_{\mathcal{M}}$. This defines a map $f \mapsto \alpha_f$ from the group G to the group of the *-automorphisms on the unital C*-algebra $\mathcal{A}_{\mathcal{M}}$ (the algebra of observables). In order to recognize this map as a representation of the group G , we must still check that $\alpha_{f_1 \circ f_2} = \alpha_{f_1} \circ \alpha_{f_2}$ for each $f_1, f_2 \in G$. Fix f_1 and f_2 in G . From covariant functoriality we deduce $\mathcal{A}(f_1 \circ f_2) = \mathcal{A}(f_1) \circ \mathcal{A}(f_2)$. For an arbitrary $a \in \mathcal{A}_{\mathcal{M}}$ we obtain

$$\alpha_{f_1 \circ f_2} a = \mathcal{A}(f_1 \circ f_2) a = \mathcal{A}(f_1)(\mathcal{A}(f_2) a) = \alpha_{f_1}(\alpha_{f_2} a)$$

because $\mathcal{A}(f_2) a \in \mathcal{A}_{\mathcal{M}}$ and therefore $\alpha_{f_1 \circ f_2} = \alpha_{f_1} \circ \alpha_{f_2}$ actually holds for each $f_1, f_2 \in G$.

We have already faced the problem of local commutativity when we gave an interpretation of the causality condition in terms of local observables. Anyway we briefly recollect the proof here for completeness. For this scope assume that \mathcal{A} is causal and fix Ω and Θ in $\mathcal{K}_{\mathcal{M}}$ such that they are causally separated in \mathcal{M} . In the category **ghs** we can consider the objects $\mathcal{M}|_{\Omega}$ and $\mathcal{M}|_{\Theta}$ and the morphisms ι_{Ω}^M and ι_{Θ}^M . In the present situation we apply the causality condition (cfr. Definition 2.1.5) and we obtain

$$[\mathcal{A}(\iota_{\Omega}^M)(\mathcal{A}(\mathcal{M}|_{\Omega})), \mathcal{A}(\iota_{\Theta}^M)(\mathcal{A}(\mathcal{M}|_{\Theta}))] = \{0\},$$

which is exactly our thesis because of Definition 2.1.8.

To prove the last part of the theorem we assume that \mathcal{A} fulfils the time slice axiom. Let Σ be a spacelike (hence acausal due to [O'N83, Chap. 14, Lem. 42, p. 425]) Cauchy surface for \mathcal{M} and let S be a connected open subset of Σ such that

$D_{\mathcal{M}}(S)$ is relatively compact in M . For convenience we write D in place of $D_{\mathcal{M}}(S)$. In first place we must check that D is in $\mathcal{K}_{\mathcal{M}}$, otherwise our thesis doesn't make sense. From [FV11, Lem. A.9, p. 48] we deduce that D is an open subset of M . Now we show that D is \mathcal{M} -causally convex. Take a \mathfrak{t} -future directed g -causal curve γ in M starting from $p \in D$ and ending in $q \in D$ and assume by contradiction that γ is not entirely contained in D . Then we find a point r along γ such that there exists an inextensible \mathfrak{t} -future directed g -timelike curve γ' in M passing through r which doesn't meet S . Hence we can use proper pieces of γ and γ' to easily build an inextensible \mathfrak{t} -future directed g -causal curve in M passing through p (or otherwise q) which doesn't meet S . This undoubtedly violates the hypothesis that both p and q are in D . Therefore D is actually \mathcal{M} -causally convex. We still need to show that D is connected. Suppose that p and q are points in D . Because of the definition of D , it is not hard to find two g -causal curves γ_1 and γ_3 in M connecting respectively p to some point r and q to some point s , with r and s in S . Since trivially $S \subseteq D$, we deduce from \mathcal{M} -causally convexity that both γ_1 and γ_3 are included in D . S is connected by hypothesis and so we find a curve γ_2 connecting r and s . If we paste γ_1 , γ_2 and γ_3 we obtain a curve connecting p to q and this proves that D is actually connected. With this preparatory results and the hypothesis that D is relatively compact, we can conclude that D is an element of $\mathcal{K}_{\mathcal{M}}$ and hence the thesis makes sense. Now we take also Ω in $\mathcal{K}_{\mathcal{M}}$ such that $S \subseteq \Omega$ and we start the real proof. As usual we can consider the globally hyperbolic spacetimes $\mathcal{M}|_{\Omega}$ and $\mathcal{M}|_D$ and the morphisms ι_{Ω}^M and ι_D^M that immerse these spacetimes in \mathcal{M} . We make a useful observation: S is a Cauchy surface for $\mathcal{M}|_D$. This is seen in the following way: S is a subset of a Cauchy surface Σ for \mathcal{M} , hence each inextensible \mathfrak{t} -future directed g -timelike curve in M meets S at most once; take now an arbitrary inextensible $\mathfrak{t}|_D$ -future directed $g|_D$ -timelike curve γ in D ; in M we can extend γ to an inextensible \mathfrak{t} -future directed g -timelike curve γ' in M ; undoubtedly γ' passes through some point in D , hence we deduce that it meets S (remember that D is the Cauchy development of S in \mathcal{M}), therefore it meets S exactly once; now we restrict γ' to D and we realize that such restriction γ'' is a $\mathfrak{t}|_D$ -future directed $g|_D$ -timelike curve in D that meets S exactly once and extends γ ; but γ was inextensible by our assumption, hence γ and γ'' coincide so that γ meets S exactly once, proving that S is a Cauchy surface for $\mathcal{M}|_D$. To proceed we introduce the subset $\Theta = \Omega \cap D$. We realize at once that Θ is an open subset of M . Furthermore we see that $\overline{\Theta} \subseteq \overline{\Omega}$, hence Θ is relatively compact in M since both Ω is such. If we take a \mathfrak{t} -future directed g -causal curve γ in M starting at $p \in \Theta$ and ending at $q \in \Theta$, we recognize that γ must be included in both Ω and D because they are \mathcal{M} -causally convex. This implies that Θ is \mathcal{M} -causally convex too. Now pick too arbitrary points p and q of Θ . Since p and q fall in D , it is easy to find two \mathfrak{t} -future directed g -causal curves γ_1 and γ_3 in M connecting respectively the point p to some point $r \in S$ and the point

q to some point $s \in S$. By hypothesis $S \subseteq \Omega$, hence also $S \subseteq \Theta$. Then both γ_1 and γ_3 are contained in Θ as a consequence of \mathcal{M} -causal convexity. S is connected by hypothesis and so we find γ_2 (automatically included in Θ) connecting r and s . Then pasting γ_1 , γ_2 and γ_3 , we connect p and q , therefore Θ is also connected. With this we have shown that $\Theta \in \mathcal{K}_{\mathcal{M}}$. Essentially this is the situation: We have a globally hyperbolic spacetime $\mathcal{M}|_D$ with a Cauchy surface S included in $\Theta \in \mathcal{K}_{\mathcal{M}}$, with $\Theta \subseteq D$. Being \mathcal{M} -causally convex, Θ is also $\mathcal{M}|_D$ -causally convex and so we can consider both the globally hyperbolic spacetimes $\mathcal{M}|_{\Theta}$ and $\mathcal{M}|_D|_{\Theta}$. We realize immediately that $\mathcal{M}|_D|_{\Theta} = \mathcal{M}|_{\Theta}$ and so the morphism ι_{Θ}^D immerses $\mathcal{M}|_{\Theta}$ in $\mathcal{M}|_D$. As we said above, the image $\iota_{\Theta}^D(\Theta) = \Theta$ includes the Cauchy surface S for $\mathcal{M}|_D$. Then it is possible to apply the time slice axiom obtaining

$$\mathcal{A}(\iota_{\Theta}^D)(\mathcal{A}(\mathcal{M}|_{\Theta})) = \mathcal{A}(\mathcal{M}|_D).$$

There is still another morphism of \mathbf{ghs} at our disposal: $\iota_{\Theta}^M \in \mathbf{Mor}_{\mathbf{ghs}}(\mathcal{M}|_{\Theta}, \mathcal{M})$. It is easy to check that $\iota_{\Theta}^M = \iota_D^M \circ \iota_{\Theta}^D$ and hence, via covariant functoriality, we deduce $\mathcal{A}(\iota_{\Theta}^M) = \mathcal{A}(\iota_D^M) \circ \mathcal{A}(\iota_{\Theta}^D)$. Applying $\mathcal{A}(\iota_D^M)$ to both sides of our last equation, we get

$$\mathcal{A}_{\mathcal{M}}(\Theta) = \mathcal{A}(\iota_{\Theta}^M)(\mathcal{A}(\mathcal{M}|_{\Theta})) = \mathcal{A}_{\mathcal{M}}(D).$$

Remembering the inclusion $\Theta \subseteq \Omega$ and applying isotony, we conclude the proof:

$$\mathcal{A}_{\mathcal{M}}(\Omega) \supseteq \mathcal{A}_{\mathcal{M}}(\Theta) = \mathcal{A}_{\mathcal{M}}(D).$$

□

Remark 2.1.10. We warn the reader that one of the properties required by the Haag-Kastler axioms is not included in our theorem, specifically we did not show that the unital C^* -algebra $\mathcal{A}_{\mathcal{M}}$ is primitive, i.e. there exists a faithful irreducible representation of $\mathcal{A}_{\mathcal{M}}$ on a Hilbert space. Hence the conclusion that the Haag-Kastler framework is completely recovered via the last theorem is not correct at all. Anyway, we will see later that the concrete locally covariant quantum field theories that we construct satisfy also this property (see the upcoming Remark 2.2.9).

2.2 Construction of a locally covariant quantum field theory

In this section we deal with the problem of building concrete locally covariant quantum field theories for situations of physical interest. In the first part we will show a procedure that leads to the construction of a causal LCQFT fulfilling the time slice axiom starting from the wave equation of a classical field represented by a section in an arbitrary vector bundle over some globally hyperbolic spacetime. To do

this we will need to specialize some more the category \mathbf{ghs} of globally hyperbolic spacetimes. As a matter of fact Definition 2.1.1 contains all the knowledge that is required to state the generally covariant locality principle without specifying anything about the physical problem to which we want to apply such principle, except the fact that it takes place over a globally hyperbolic spacetime. This is the approach followed by Brunetti, Fredenhagen and Verch in [BFV03] when they proposed the GCLP, as well as by other authors even in the more recent papers on this topic (e.g. [FV11, Dap11]). This choice is done to underline that the consequences of the GCLP (specifically Theorem 2.1.9 in the present case) do not depend upon any of the properties of the specific quantum field that one may consider, except for the fact that it is set over a globally hyperbolic spacetime. This is one of the strong points of the GCLP.

Following Fewster and Verch [FV11], we could even enlarge the category \mathbf{alg} in order to take into account a very wide range of physical situations (not only quantum fields, but also classical dynamical systems too, such as classical fields or mechanical systems). There is not a precise way to define the new target category: which is the more convenient setting for a theory actually depends upon the type of physical problem the theory deals with (e.g. C^* -algebras for quantum fields and symplectic spaces for classical fields). The key point is that there exists a common mathematical framework in which it is possible to formulate all those theories: they are recognized to be covariant functors from the category \mathbf{ghs} (eventually with some more data concerning the specific problem under consideration) to a convenient category that fit the physical problem in the best way. It is the functorial approach that unifies all these physical theories and for all of them it is possible to speak of causality and time slice axiom, although this must be done in a way that is adapted to the mathematical framework chosen to describe the physical system we are interested in.

During the construction of a LCQFT we will encounter a relevant example of what we are saying. Specifically, we will see that a classical field is comfortably described by a covariant functor from \mathbf{ghs} (with some structure that pertains to the field itself, essentially the wave equation governing its dynamics) to the category having symplectic spaces as objects and symplectic maps as morphisms.

2.2.1 From classical field theory...

We want to describe a classical field over some d -dimensional globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ that is modeled by a smooth section u in a vector bundle E of rank n over M satisfying the normally hyperbolic equation $Au = 0$ on each point of M , where E is endowed with an inner product denoted by $\overset{E}{\cdot}$ and A is a formally selfadjoint normally hyperbolic operator on E over \mathcal{M} .

As we anticipated above, we are going to build a functor from a slightly modified

version of \mathfrak{ghs} to a proper category that we define right now.

Categories

Definition 2.2.1. The category \mathfrak{ghs}^f is defined in the following way:

- objects are triples (\mathcal{M}, E, A) , where $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ is a d -dimensional globally hyperbolic spacetime, E is a vector bundle of rank n over M endowed with an inner product denoted by \cdot^E and A is a formally selfadjoint normally hyperbolic operator on E over \mathcal{M} ;
- morphisms between two arbitrary objects (\mathcal{M}, E, A) and (\mathcal{N}, F, B) are vector bundle homomorphisms (ψ, Ψ) compatible with the inner products \cdot^E and \cdot^F and the formally selfadjoint normally hyperbolic operators A and B (we will explain the meaning of these conditions immediately after this definition), where ψ is a morphism of \mathfrak{ghs} from \mathcal{M} to \mathcal{N} ;
- the composition law is the ordinary composition of vector bundle homomorphisms, i.e. the composition of function for both members of the pairs.

We also define \mathfrak{ssp} as the category whose objects are symplectic spaces (V, ω) , whose morphisms between two arbitrary objects (V, σ) and (W, ω) are symplectic maps ξ and whose composition law is the usual composition of functions.

Before proceeding, we want to specify the meaning of the condition of compatibility with the inner products and with the normally hyperbolic operators that is required to the morphisms of \mathfrak{ghs}^f . This completes the definition of \mathfrak{ghs}^f . We also take the chance to underline some particular properties of the vector bundle homomorphisms we are going to deal with.

The condition of compatibility with inner products means that each vector bundle homomorphism that we take into account must be fiberwise an isometry with respect to the vector space non degenerate inner products induced on each fiber by the inner products on the vector bundles. To be precise, we require that each vector bundle homomorphisms (ψ, Ψ) from (\mathcal{M}, E, A) to (\mathcal{N}, F, B) satisfies the following condition:

$$(\Psi_p \mu) \cdot_{\psi(p)}^F (\Psi_p \nu) = \mu \cdot_p^E \nu$$

for each $p \in M$ and each $\mu, \nu \in E_p$, where M is the manifold underlying \mathcal{M} . This ensures that Ψ is fiberwise isometric, hence, in particular, Ψ_p is an injective vector space homomorphism for each $p \in M$ because of the non degeneracy of inner products. This observation has a relevant consequence: for each $p \in M$

$$n = \dim E_p = \dim (\Psi_p (E_p)) \leq \dim F_{\psi(p)} = n.$$

This fact implies that Ψ_p is a vector space isomorphism for each $p \in M$ (however Ψ is not a vector bundle isomorphism unless ψ is bijective).

The condition of compatibility with the formally selfadjoint normally hyperbolic operators is slightly more involved. First of all notice that we are in position to apply Remark 1.1.15: ψ is an embedding whose image is an open subset of its codomain and Ψ is fiberwise a vector space isomorphism. Then we obtain the new vector bundle $\Psi(E)$ of rank n over the d -dimensional manifold $\psi(M)$ and the vector bundle isomorphism (ψ', Ψ') obtained from the restriction of (ψ, Ψ) to $\Psi(E)$. Now we take $u \in \mathcal{D}(M, E)$ and, using the vector bundle isomorphism (ψ', Ψ') and Remark 1.1.17, we introduce the section

$$u' = \Psi' \circ u \circ \psi'^{-1} : \psi(M) \rightarrow \Psi(E).$$

Since ψ' is a homeomorphism, it holds that

$$\text{supp}(u') = \psi'(\text{supp}(u))$$

and so it turns out that u' is a compactly supported section because $\text{supp}(u)$ is compact in M . Using the fact that u' is null outside a compact subset of $\psi(M)$, we can define the smooth compactly supported section $u'' : N \rightarrow F$ via the formula

$$u''(q) = \begin{cases} u'(q) & \text{if } q \in \psi(M), \\ 0 & \text{if } q \in N \setminus \psi(M). \end{cases}$$

This defines a map between the vector spaces $\mathcal{D}(M, E)$ and $\mathcal{D}(N, F)$:

$$\begin{aligned} \text{ext}_\Psi : \mathcal{D}(M, E) &\rightarrow \mathcal{D}(N, F) \\ u &\mapsto u''. \end{aligned} \tag{2.2.1}$$

Notice that such map is trivially linear and that it transforms the support through ψ :

$$\text{supp}(\text{ext}_\Psi u) = \psi(\text{supp}(u)).$$

This construction was made to be in a position that allows us to correctly state the condition of compatibility with the normally hyperbolic operators A and B : for each $u \in \mathcal{D}(M, E)$ it holds that

$$\text{ext}_\Psi(Au) = B(\text{ext}_\Psi u).$$

Till now, we have used (ψ', Ψ') simply to define the extension map ext_Ψ . However such map could be also defined directly using simply ψ' and Ψ . The real reason that prompted us to the construction of (ψ', Ψ') is that it gives us the opportunity to

build a new object of \mathbf{ghs}^f . Consider the globally hyperbolic spacetime $\psi(\mathcal{M})$ (cfr. Remark 2.1.2) and the vector bundle $\Psi(E)$. On $\Psi(E)$ we put the restriction of the inner product of F as inner product and automatically we find that Ψ' is fiberwise an isometry. We define A_Ψ on in a way that Ψ' automatically satisfies the condition of compatibility with A_Ψ and B : A_Ψ is the linear operator from $C^\infty(\psi(M), \Psi(E))$ to itself defined by the formula

$$A_\Psi u = \Psi' \circ (A(\Psi'^{-1} \circ u \circ \psi')) \circ \psi'^{-1} \quad \forall u \in C^\infty(\psi(M), \Psi(E)). \quad (2.2.2)$$

In this way A_Ψ is well defined because (ψ', Ψ') is a vector bundle isomorphism (see Remark 1.1.17) and one can easily check that A_Ψ is a formally selfadjoint normally hyperbolic operator on $\Psi(E)$ over $\psi(\mathcal{M})$ (such properties are directly inherited from the same properties that are known to hold for A). As we anticipated, the definition of A_Ψ is given in such a way that automatically (ψ', Ψ') becomes compatible with A and A_Ψ : noting that

$$(\text{ext}_{\Psi'} u)(q) = \begin{cases} (\Psi' \circ u \circ \psi'^{-1})(q) & \text{if } q \in \psi(M), \\ 0 & \text{if } q \in \psi(M) \setminus \psi(M) \end{cases} = (\Psi' \circ u \circ \psi'^{-1})(q), \quad (2.2.3)$$

for each $u \in \mathcal{D}(M, E)$ and each $q \in \psi(M)$ and choosing $u = \text{ext}_{\Psi'} v$ for any $v \in \mathcal{D}(M, E)$ in eq. (2.2.2), we read

$$A_\Psi(\text{ext}_{\Psi'} v) = \text{ext}_{\Psi'}(Av)$$

for each $v \in \mathcal{D}(M, E)$. Then we have built the object $(\psi(\mathcal{M}), \Psi(E), A_\Psi)$ of \mathbf{ghs}^f . Since ψ' is a morphism of \mathbf{ghs} from \mathcal{M} to $\psi(\mathcal{M})$ whose inverse is also a morphism, we recognize (ψ', Ψ') to be a morphism of \mathbf{ghs}^f from (\mathcal{M}, E, A) to $(\psi(\mathcal{M}), \Psi(E), A_\Psi)$ whose inverse (ψ'^{-1}, Ψ'^{-1}) is a morphism too. The situation of eq. (2.2.3) holds whenever we deal with a vector bundle isomorphism, in particular we can define similarly $\text{ext}_{\Psi'^{-1}}$.

In Remark 1.1.14 we showed that there is also a vector bundle homomorphism $(\iota_{\psi(M)}^N, \iota_{\Psi(E)}^F)$ from $\Psi(E)$ to F . We already know from our discussion on the category \mathbf{ghs} that $\iota_{\psi(M)}^N$ is a morphism of \mathbf{ghs} from $\psi(\mathcal{M})$ to \mathcal{N} . If we show that $(\iota_{\psi(M)}^N, \iota_{\Psi(E)}^F)$ is compatible with the inner products of $\Psi(E)$ and F and with the normally hyperbolic operators A_Ψ and B , we can conclude that $(\iota_{\psi(M)}^N, \iota_{\Psi(E)}^F)$ is a morphism of \mathbf{ghs}^f . Both requirements actually hold because of the definitions of $(\iota_{\psi(M)}^N, \iota_{\Psi(E)}^F)$, of the inner product on $\Psi(E)$ as the restriction of the inner product of F and of the normally hyperbolic operator A_Ψ . We explicitly check the compatibility with A_Ψ and B . Notice that for each $v \in \mathcal{D}(\psi(M), \Psi(E))$ and each

$q \in \psi(M)$

$$\begin{aligned} \left(\text{ext}_{\iota_{\Psi(E)}^F} v \right) (q) &= \begin{cases} \left(\left(\iota_{\Psi(E)}^F \right)' \circ v \circ \left(\iota_{\psi(M)}^N \right)^{-1} \right) (q) & \text{if } q \in \psi(M), \\ 0 & \text{if } q \in N \setminus \psi(M) \end{cases} \\ &= \begin{cases} v(q) & \text{if } q \in \psi(M), \\ 0 & \text{if } q \in N \setminus \psi(M). \end{cases} \end{aligned}$$

For each $v \in \mathcal{D}(\psi(M), \Psi(E))$, exploiting eq. (2.2.3), we find

$$\text{ext}_{\iota_{\Psi(E)}^F} (A_{\Psi} v) = \text{ext}_{\iota_{\Psi(E)}^F} (A_{\Psi} (\text{ext}_{\Psi'} (\text{ext}_{\Psi'^{-1}} v))) = \text{ext}_{\iota_{\Psi(E)}^F} (\text{ext}_{\Psi'} (A (\text{ext}_{\Psi'^{-1}} v))).$$

For each $u \in \mathcal{D}(M, E)$ and each $q \in \psi(M)$ we also have

$$\begin{aligned} \left(\text{ext}_{\iota_{\Psi(E)}^F} (\text{ext}_{\Psi'} u) \right) (q) &= \begin{cases} (\Psi' \circ u \circ \psi'^{-1})(q) & \text{if } q \in \psi(M), \\ 0 & \text{if } q \in N \setminus \psi(M) \end{cases} \\ &= (\text{ext}_{\Psi'} u)(q). \end{aligned}$$

We insert our last equation in the previous one, we exploit the compatibility property of (ψ, Ψ) with A and B and we recall the definitions of ext_{Ψ} and $\text{ext}_{\iota_{\Psi(E)}^F}$. In this way we obtain

$$\text{ext}_{\iota_{\Psi(E)}^F} (A_{\Psi} v) = \text{ext}_{\Psi} (A (\text{ext}_{\Psi'^{-1}} v)) = B (\text{ext}_{\Psi} (\text{ext}_{\Psi'^{-1}} v)) = B \left(\text{ext}_{\iota_{\Psi(E)}^F} v \right)$$

for each $v \in \mathcal{D}(\psi(M), \Psi(E))$. This shows that $(\iota_{\psi(M)}^N, \iota_{\Psi(E)}^F)$ is compatible with A_{Ψ} and B and hence it is actually a morphism of \mathbf{ghs}^f . Using $(\iota_{\psi(M)}^N, \iota_{\Psi(E)}^F)$ and (ψ', Ψ') , we can decompose the original morphism (ψ, Ψ) according the formula

$$(\psi, \Psi) = (\iota_{\psi(M)}^N, \iota_{\Psi(E)}^F) \circ (\psi', \Psi').$$

Having explicated the meaning of all the requirements in Definition 2.2.1, we can ask whether \mathbf{ghs}^f and \mathbf{ssp} are actually categories. This question is faced in the subsequent remark.

Remark 2.2.2. We check that \mathbf{ghs}^f is actually a category. The first thing to be done is to verify that the composition law is well defined. To this end consider three objects (\mathcal{M}, E, A) , (\mathcal{N}, F, B) and (\mathcal{O}, G, C) , a morphism (ϕ, Φ) from (\mathcal{M}, E, A) to (\mathcal{N}, F, B) and a morphism (ψ, Ψ) from (\mathcal{N}, F, B) to (\mathcal{O}, G, C) . As we have seen in Remark 2.1.2, $\psi \circ \phi$ is still a morphism of \mathbf{ghs} . Since $\Phi : E \rightarrow F$ and $\Psi : F \rightarrow G$ are smooth maps, using coordinate charts of the manifolds E, F, G we immediately realize that also $\Psi \circ \Phi : E \rightarrow G$ is a smooth map. Let π_E, π_F and π_G be the projection maps respectively of E, F and G . We know that $\pi_F \circ \Phi = \phi \circ \pi_E$ and that

$\pi_G \circ \Psi = \psi \circ \pi_F$. Then, applying the associativity of the composition of functions, we find

$$\pi_G \circ (\Psi \circ \Phi) = \psi \circ \pi_F \circ \Phi = (\psi \circ \phi) \circ \pi_E.$$

As for fiberwise linearity, consider a point $p \in M$. Taking into account the map

$$\begin{aligned} (\Psi \circ \Phi)_p : E_p &\rightarrow G_{\psi(\phi(p))} \\ \mu &\mapsto (\Psi \circ \Phi) \mu, \end{aligned}$$

we can easily check that

$$(\Psi \circ \Phi)_p = \Psi_{\phi(p)} \circ \Phi_p$$

and hence $(\Psi \circ \Phi)_p$ is linear being the composition of linear maps. This shows that $(\psi, \Psi) \circ (\phi, \Phi) = (\psi \circ \phi, \Psi \circ \Phi)$ is a vector bundle homomorphism from E to G . We check its compatibility with the inner products of the vector bundles involved exploiting the same property that is assumed to hold for both (ϕ, Φ) and (ψ, Ψ) : for each $p \in M$ and each $\mu, \nu \in E_p$ we have

$$\left((\Psi \circ \Phi)_p \mu \right)^{G_{\psi(\phi(p))}} \cdot \left((\Psi \circ \Phi)_p \nu \right) = (\Phi_p \mu)^{F_{\phi(p)}} \cdot (\Phi_p \nu) = \mu^E \cdot_p \nu.$$

As for the compatibility with the normally hyperbolic operators, for each $u \in \mathcal{D}(M, E)$ it holds that

$$\text{ext}_\Psi (\text{ext}_\Phi (Au)) = \text{ext}_\Psi (B(\text{ext}_\Phi u)) = C(\text{ext}_\Psi (\text{ext}_\Phi u)), \quad (2.2.4)$$

where $\text{ext}_\Phi : \mathcal{D}(M, E) \rightarrow \mathcal{D}(N, F)$ and $\text{ext}_\Psi : \mathcal{D}(N, F) \rightarrow \mathcal{D}(O, G)$ are the extension maps obtained applying the discussion that led to eq. (2.2.1) to (ϕ, Φ) with $\phi(M)$ and respectively to (ψ, Ψ) with $\psi(N)$. In the same way from $(\psi, \Psi) \circ (\phi, \Phi)$ with $(\psi \circ \phi)(M)$, we obtain the extension map $\text{ext}_{\Psi \circ \Phi} : \mathcal{D}(M, E) \rightarrow \mathcal{D}(O, E)$. Our scope now is to show that $\text{ext}_{\Psi \circ \Phi} = \text{ext}_\Psi \circ \text{ext}_\Phi$: for each $u \in \mathcal{D}(M, E)$ and each $r \in O$

$$\begin{aligned} (\text{ext}_{\Psi \circ \Phi} u)(r) &= \begin{cases} ((\Psi \circ \Phi)' \circ u \circ (\psi \circ \phi)^{\prime -1})(r) & \text{if } r \in (\psi \circ \phi)(M), \\ 0 & \text{if } r \in O \setminus (\psi \circ \phi)(M) \end{cases} \\ &= \begin{cases} (\Psi \circ \Phi)'(u((\psi \circ \phi)^{\prime -1}(r))) & \text{if } r \in (\psi \circ \phi)(M), \\ 0 & \text{if } r \in O \setminus (\psi \circ \phi)(M) \end{cases} \\ &= \begin{cases} \Psi'(\Phi'(u(\psi^{\prime -1}(\phi^{\prime -1}(r)))) & \text{if } r \in (\psi \circ \phi)(M), \\ 0 & \text{if } r \in O \setminus (\psi \circ \phi)(M), \end{cases} \end{aligned}$$

while

$$\begin{aligned}
 (\text{ext}_\Psi(\text{ext}_\Phi u))(r) &= \begin{cases} (\Psi' \circ (\text{ext}_\Phi u) \circ \psi'^{-1})(r) & \text{if } r \in \psi(N), \\ 0 & \text{if } r \in O \setminus \psi(N) \end{cases} \\
 &= \begin{cases} \Psi'(\Phi'(u(\phi'^{-1}(\psi'^{-1}(r)))))) & \text{if } r \in \psi(\phi(M)), \\ 0 & \text{if } r \in \psi(N) \setminus \psi(\phi(N)), \\ 0 & \text{if } r \in O \setminus \psi(N), \end{cases}
 \end{aligned}$$

hence the equation

$$\text{ext}_{\Psi \circ \Phi} = \text{ext}_\Psi \circ \text{ext}_\Phi \quad (2.2.5)$$

holds as expected. Inserting such equation in eq. (2.2.4), we conclude that for each $u \in \mathcal{D}(M, E)$

$$\text{ext}_{\Psi \circ \Phi}(Au) = B(\text{ext}_{\Psi \circ \Phi} u).$$

Then $(\psi, \Psi) \circ (\phi, \Phi)$ is actually a morphism of \mathbf{ghs}^f . This proves that the composition law is well defined. To conclude, we have to check the category axioms. If we take an arbitrary object (\mathcal{M}, E, A) , the identity law is satisfied by the vector bundle homomorphism $\text{id}_{(\mathcal{M}, E, A)} = (\text{id}_M, \text{id}_E)$, where $\text{id}_M : M \rightarrow M, p \mapsto p$ and $\text{id}_E : E \rightarrow E, \mu \mapsto \mu$. The associativity of the composition law is trivial because this property is inherited from the associativity of the ordinary composition of functions.

Turning our attention to \mathbf{ssp} , we take three objects (U, ρ) , (V, σ) and (W, ω) , a morphism ξ from (U, ρ) to (V, σ) and a morphism η from (V, σ) to (W, ω) and we consider the function $\eta \circ \xi : U \rightarrow W$. We obtain a linear map between the vector spaces U and W that preserves the symplectic forms because both ξ and η are symplectic maps:

$$\omega(\eta(\xi v), \eta(\xi w)) = \sigma(\xi v, \xi w) = \rho(v, w) \quad \forall v, w \in U.$$

Hence we realize that $\eta \circ \xi$ is a morphism from (U, ρ) to (W, ω) and this proves that the composition law is well defined. For each object (V, σ) , the identity law is satisfied by the morphism $\text{id}_{(V, \sigma)}$ from (V, σ) to itself defined by $\text{id}_{(V, \sigma)} v = v$ for each $v \in V$. Also in this case the associativity of the composition law follows from same property of the composition of functions.

Functor

We begin now the construction of a covariant functor that maps each object of \mathbf{ghs}^f to an object of \mathbf{ssp} . In our intention the object of \mathbf{ghs}^f should describe the physical problem that we deal with (in this case a wave equation describing a field over a globally hyperbolic spacetime) and the corresponding object of \mathbf{ssp} should be the solution for such problem (all the dynamical configurations of the field, i.e. the

solutions of all the Cauchy problems with compactly supported initial data related to the wave equation mentioned above). We may say that the object of \mathbf{ghs}^f describes the system we want to study, while the corresponding object of \mathbf{ssp} contains all the knowledge about the dynamics of that system. In the following we will make extensive use of the results in Subsection 1.3.4.

Assume that we are given an object (\mathcal{M}, E, A) of \mathbf{ghs}^f . Applying Corollary 1.3.16, we obtain the advanced and retarded Green operators e_A^a and e_A^r for A . With them we can introduce the causal propagator $e_A = e_A^a - e_A^r$ for A and Corollary 1.3.19 tells us that the space of the solutions for all the homogeneous Cauchy problems with compactly supported initial data associated to A is given by the vector space $V = e_A(\mathcal{D}(M, E))$. Moreover Proposition 1.3.20 provides an important information on the structure of the kernel of the causal propagator e_A , precisely $\ker e_A = A(\mathcal{D}(M, E))$. We also notice that in the current situation A is supposed to be formally selfadjoint, i.e. $A^* = A$ (cfr. Remark 1.3.11). It follows in particular that A^* is a normally hyperbolic operator too and that its advanced and retarded Green operators are exactly e_A^a and e_A^r . Hence Proposition 1.3.21 in the present situation means that e_A^a and e_A^r are the formally adjoints of each other. As a consequence of this fact, we see that e_A is formally antiselfadjoint. All these observations will be exploited soon.

Lemma 2.2.3. *Consider the situation presented above and bear in mind that $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$. Taking into account the vector space $V = e_A(\mathcal{D}(M, E))$, the map*

$$\begin{aligned} \sigma : V \times V &\rightarrow \mathbb{R} \\ (u, v) &\mapsto \int_M \left((e_A f) \cdot^E h \right) d\mu_g, \end{aligned}$$

where f and h in $\mathcal{D}(M, E)$ are such that $e_A f = u$ and $e_A h = v$ and $d\mu_g$ is the standard volume form on \mathcal{M} , is well defined, bilinear, antisymmetric and non degenerate, i.e. σ is a symplectic form on V and hence (V, σ) is a symplectic space, as a matter of fact an object of \mathbf{ssp} .

Proof. We check that σ is well defined. Fix u and v in $V = e_A(\mathcal{D}(M, E))$ and take f_1, f_2, h_1, h_2 in $\mathcal{D}(M, E)$ such that $e_A f_1 = u = e_A f_2$ and $e_A h_1 = v = e_A h_2$. Exploiting twice the antiselfadjointness of e_A , we deduce that

$$\begin{aligned} \int_M \left((e_A f_1) \cdot^E h_1 \right) d\mu_g &= \int_M \left((e_A f_2) \cdot^E h_1 \right) d\mu_g = - \int_M \left(f_2 \cdot^E (e_A h_1) \right) d\mu_g \\ &= - \int_M \left(f_2 \cdot^E (e_A h_2) \right) d\mu_g = \int_M \left((e_A f_2) \cdot^E h_2 \right) d\mu_g. \end{aligned}$$

Till this point we have shown that σ is well defined. Bilinearity follows directly from the linearity of the causal propagator, fiberwise bilinearity of the inner product

in E and the linearity of the integral. As for antisymmetry, we take u and v in V . Then we find f and h in $\mathcal{D}(M, E)$ such that $e_A f = u$ and $e_A h = v$. With this ingredients we evaluate $\sigma(u, v)$ bearing in mind that e_A is antiselfadjoint and that the inner product of E is fiberwise symmetric:

$$\begin{aligned}\sigma(u, v) &= \int_M \left((e_A f)^E \cdot h \right) d\mu_g = - \int_M \left(f^E \cdot (e_A h) \right) d\mu_g = - \int_M \left((e_A h)^E \cdot f \right) d\mu_g \\ &= -\sigma(v, u).\end{aligned}$$

We are left with the proof of non degeneracy. Suppose that we have $u \in V$ such that $\sigma(u, v) = 0$ for each $v \in V$. This means that

$$\int_M \left(u^E \cdot f \right) d\mu_g = 0$$

for each $f \in \mathcal{D}(M, E)$. But this implies that $u = 0$. Hence σ is actually non degenerate. \square

The last theorem provides the first part of our candidate covariant functor, specifically the map

$$\begin{aligned}\mathcal{B} : \quad \text{Obj}_{\mathfrak{ghs}^f} &\rightarrow \text{Obj}_{\mathfrak{ghs}^f} \\ (\mathcal{M}, E, A) &\mapsto (V, \sigma).\end{aligned}$$

The second part should consist of a map

$$\text{Mor}_{\mathfrak{ghs}^f}((\mathcal{M}, E, A), (\mathcal{N}, F, B)) \rightarrow \text{Mor}_{\mathfrak{ssp}}(\mathcal{B}(\mathcal{M}, E, A), \mathcal{B}(\mathcal{N}, F, B))$$

for each pair of objects (\mathcal{M}, E, A) and (\mathcal{N}, F, B) of \mathfrak{ghs}^f . To build such map we need a preliminary result.

Lemma 2.2.4. *Let $(\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t}), E, A)$ and $(\mathcal{N} = (N, h, \mathfrak{p}, \mathfrak{u}), F, B)$ be two objects of \mathfrak{ghs}^f and let (ψ, Ψ) be a morphism between these objects. Denote the advanced/retarded Green operators for A and B respectively with $e_A^{a/r}$ and $e_B^{a/r}$ and consider the maps ext_Ψ defined in eq. (2.2.1) and the map*

$$\begin{aligned}\text{res}_\Psi : C^\infty(N, F) &\rightarrow C^\infty(M, E) \\ v &\mapsto \Psi'^{-1} \circ \left(v|_{\psi(M)} \right) \circ \psi' .\end{aligned}$$

Then we have $\text{res}_\Psi \circ e_B^{a/r} \circ \text{ext}_\Psi = e_A^{a/r}$.

Proof. We start showing that the map res_Ψ is well defined. Consider a section $v \in C^\infty(N, F)$. By $v|_{\psi(M)}$ we mean the function from $\psi(M)$ to $\Psi(E)$ defined by $v|_{\psi(M)}(q) = v(q)$ for each $q \in \psi(M)$. The domain and the codomain of $v|_{\psi(M)}$

are manifolds, hence we can ask whether this function is continuous and, in this case, whether it is also smooth. Both questions have positive answer because of the topologies and the atlases of $\psi(M)$ and $\Psi(E)$, which are open subsets of N and respectively $\Psi(E)$, are induced via restriction from those of N and F (cfr. Remark 1.1.7 for $\psi(M)$ and Remark 1.1.15 for $\Psi(E)$). We have recognized $v|_{\psi(M)}$ to be a smooth map from $\psi(M)$ to $\Psi(E)$. But the remark cited above tells us also that $\Psi(E)$ is a vector bundle over the manifold $\psi(M)$. Hence we can also ask whether $v|_{\psi(M)}$ is a section in $\Psi(E)$ and again we get a positive answer because $\pi_{\Psi(E)}$ is defined as the restriction of π_F :

$$\pi_{\Psi(E)} \left(v|_{\psi(M)}(q) \right) = \pi_F(v(q)) = q \quad \forall q \in \psi(M).$$

From Remark 1.1.17 applied to the section $v|_{\psi(M)}$ and the vector bundle isomorphism (ψ', Ψ') , we finally obtain the section in E we were looking for:

$$\Psi'^{-1} \circ \left(v|_{\psi(M)} \right) \circ \psi'.$$

This proves that the definition of res_Ψ makes sense.

Our strategy to prove the thesis of this lemma consists in showing that $\text{res}_\Psi \circ e_B^{a/r} \circ \text{ext}_\Psi$ is an advanced/retarded Green operator for A (cfr. Definition 1.3.15), so that it must coincide with $e_A^{a/r}$ because Corollary 1.3.16 assures uniqueness. We consider only the case of the advanced Green operator, the other case being similar.

To show that $\text{res}_\Psi \circ e_B^a \circ \text{ext}_\Psi$ is the advanced Green operator for A we do not check that it fulfils the requirements of Definition 1.3.15, but we prefer to show that it generates exactly one fundamental solution $U_A^r(p)$ for A^* (in this case $A^* = A$, but we will not use such property) with \mathcal{M} -future compact support for each point p in M according to the formula

$$U_A^r(p)[u] = ((\text{res}_\Psi \circ e_B^a \circ \text{ext}_\Psi)u)(p) \quad \forall u \in \mathcal{D}(M, E).$$

If we succeed in our scope, via Corollary 1.3.16 we obtain an advanced Green operator for A from the collection of fundamental solutions with future compact support $\{U_A^r(p) : p \in M\}$ for A^* . This operator is defined through a formula identical to the one written above, but intended in the opposite sense, hence we find that $\text{res}_\Psi \circ e_B^a \circ \text{ext}_\Psi$ is exactly this operator. In particular it follows that $\text{res}_\Psi \circ e_B^a \circ \text{ext}_\Psi$ is an advanced Green operator for A .

Now we fix $p \in M$. Together with the normally hyperbolic operator A^* (normal hyperbolicity of A^* follows from the hypothesis that A is normally hyperbolic even if $A^* \neq A$), we consider also its distributional version (still denoted by A^*) as it is defined following the procedure shown in Remark 1.3.10 using E_p as target vector

space for the space of distributions (see the discussion before Definition 1.3.12):

$$A^* : \mathcal{D}'(M, E^*, E_p) \rightarrow \mathcal{D}'(M, E^*, E_p)$$

Indeed the hypothesis of formal selfadjointness implies that $A = A^*$ also for the operators in distributional sense (cfr. Remark 1.3.11), but this fact is not necessary for our conclusions and hence in this proof we will distinguish between A^* and A as it would have been without the hypothesis of formal selfadjointness. We note that for each $u \in \mathcal{D}(M, E)$ it holds

$$(A^* U_A^r(p)) [u] = U_A^r(p) [Au] = ((\text{res}_\Psi \circ e_B^a \circ \text{ext}_\Psi) Au)(p) = u(p) = \delta_p [u]$$

due to the compatibility of (ψ, Ψ) with A and B and the fact that e_B^a is the advanced Green operator for B . This means that $U_A^r(p)$ is a fundamental solution for A^* at p (cfr. Definition 1.3.12). The support of the distribution $U_A^r(p)$ is given by

$$\text{supp}(U_A^r(p)) = \left\{ q \in M : \begin{array}{l} \text{for each neighborhood } U \text{ of } q \text{ in } M \text{ there} \\ \text{exists a section } u \in \mathcal{D}(M, E) \text{ with support} \\ \text{included in } U \text{ such that } U_A^r(p) [u] \neq 0 \end{array} \right\}.$$

But $U_A^r(p) [u] \neq 0$ means that $(e_B^a(\text{ext}_\Psi u))(\psi(p)) \neq 0$, that is $U_B^r(\psi(p)) [\text{ext}_\Psi u] \neq 0$, where $U_B^r(\psi(p))$ is the unique fundamental solution for B^* at $\psi(p)$ with future compact support generated by the advanced Green operator e_B^a for B according to Corollary 1.3.16. On the one hand we deduce that

$$\text{supp}(U_A^r(p)) \subseteq \psi^{-1} \left(\left\{ q \in \psi(M) : \begin{array}{l} \text{for each neighborhood } V \text{ of } q \text{ in } N \\ \text{there exists a section } v \in \mathcal{D}(N, F) \\ \text{with support included in } V \text{ such} \\ \text{such that } U_B^r(\psi(p)) [v] \neq 0 \end{array} \right\} \right).$$

On the other hand the support of $U_B^r(\psi(p))$ is of the form

$$\text{supp}(U_B^r(\psi(p))) = \left\{ q \in N : \begin{array}{l} \text{for each neighborhood } V \text{ of } q \text{ in } N \text{ there} \\ \text{exists a section } v \in \mathcal{D}(N, F) \text{ with support} \\ \text{included in } V \text{ such that } U_B^r(\psi(p)) [v] \neq 0 \end{array} \right\}.$$

From the comparison of the last two equations we conclude that

$$\text{supp}(U_A^r(p)) \subseteq \psi^{-1}(\text{supp}(U_B^r(\psi(p)))).$$

Theorem 1.3.14 gives us an important information about the support of $U_B^r(\psi(p))$, namely the inclusion

$$\text{supp}(U_B^r(\psi(p))) \subseteq J_-^{\mathcal{A}}(\psi(p)),$$

therefore we obtain

$$\text{supp}(U_A^r(p)) \subseteq \psi^{-1}(J_-^{\mathcal{N}}(\psi(p))).$$

Consider now a point $q \in \psi^{-1}(J_-^{\mathcal{N}}(\psi(p)))$. We recognize that $\psi(p)$ and $\psi(q)$ are both in $\psi(M)$ and moreover $\psi(q)$ falls in $J_-^{\mathcal{N}}(\psi(p))$. Hence we find a \mathbf{u} -past directed h -causal curve γ in N starting at $\psi(p)$ and ending at $\psi(q)$. Since we assumed that ψ is a morphism of \mathbf{ghs} , $\psi(M)$ is \mathcal{N} -causally convex and so γ is entirely included in $\psi(M)$. Then we can consider the curve $\psi'^{-1} \circ \gamma$. Since ψ is isometric and preserves time orientation, we deduce that $\psi'^{-1} \circ \gamma$ is a \mathbf{t} -past directed g -causal curve in M starting from p and ending in q . This implies that $q \in J_-^{\mathcal{M}}(p)$. Then we have the inclusion

$$\text{supp}(U_A^r(p)) \subseteq J_-^{\mathcal{M}}(p).$$

By assumption, \mathcal{M} is a globally hyperbolic spacetime and so $J_-^{\mathcal{M}}(p) \cap J_+^{\mathcal{M}}(q)$ is a compact subset of M for each $q \in M$. Hence $J_-^{\mathcal{M}}(p)$ is \mathcal{M} -future compact and then $\text{supp}(U_A^r(p))$ is \mathcal{M} -future compact too being a closed subset of $J_-^{\mathcal{M}}(p)$. This shows that $U_A^r(p)$ is a fundamental solution for A^* at p with \mathcal{M} -future compact support for each $p \in M$. Uniqueness follows from Lemma 1.3.13. Hence our strategy of proof can be carried out without difficulties and the thesis is proved. \square

Now we are ready to face the main problem, that is. the determination of a function that maps each morphism between two objects of \mathbf{ghs}^f to a morphism between the corresponding two objects of \mathbf{ssp} .

Lemma 2.2.5. *Let $(\mathcal{M} = (M, g, \mathbf{o}, \mathbf{t}), E, A)$ and $(\mathcal{N} = (M, h, \mathbf{p}, \mathbf{u}), F, B)$ be two objects of \mathbf{ghs}^f and denote with (V, σ) and (W, ω) the corresponding objects of \mathbf{ssp} provided by Lemma 2.2.3. Consider a morphism (ψ, Ψ) of \mathbf{ghs}^f from (\mathcal{M}, E, A) to (\mathcal{N}, F, B) . Denote with e_A and e_B the causal propagators for A and B respectively. Then the map*

$$\begin{aligned} \xi : V &\rightarrow W \\ u &\mapsto e_B(\text{ext}_{\Psi} f), \end{aligned}$$

where $f \in \mathcal{D}(M, E)$ is such that $e_A f = u$, is well defined, linear and compatible with σ and ω , i.e. ξ is a symplectic map from (V, σ) to (W, ω) , which is to say that ξ is a morphism of \mathbf{ssp} between the objects (V, σ) and (W, ω) .

Proof. Fix $u \in V = e_A(\mathcal{D}(M, E))$ and consider f_1 and f_2 in $\mathcal{D}(M, E)$ such that $e_A f_1 = u = e_A f_2$. In order to have ξ well defined we must show that $e_B(\text{ext}_{\Psi} f_1) = e_B(\text{ext}_{\Psi} f_2)$. Because of the linearity of the causal propagator and of the extension map (see how ext_{Ψ} was defined in eq. (2.2.1)), this is equivalent to prove that $f' = \text{ext}_{\Psi} f$ falls in the kernel of e_B , where f denotes $f_1 - f_2$. We know that $\ker e_A = A(\mathcal{D}(N, F))$ and that $f \in \ker e_A$. Hence we find $h \in \mathcal{D}(M, E)$ such that

$Ah = f$. Then, exploiting the compatibility of (ψ, Ψ) with A and B , we obtain

$$\text{ext}_\Psi f = \text{ext}_\Psi (Ah) = B(\text{ext}_\Psi h).$$

We have just found $h' = \text{ext}_\Psi h$ in $\mathcal{D}(N, F)$ such that $Bh' = f'$. This implies that f' falls in $B(\mathcal{D}(N, F)) = \ker e_B$.

Linearity of the causal propagators and of the extension map assures that ξ is linear too.

Consider now u_1 and u_2 in V and evaluate $\omega(\xi u_1, \xi u_2)$. We find f_1 and f_2 in $\mathcal{D}(M, E)$ such that $e_A f_1 = u_1$ and $e_A f_2 = u_2$. Then, exploiting the definition of ξ , we get

$$\omega(\xi u_1, \xi u_2) = \int_N \left((e_B(\text{ext}_\Psi f_1))^{\cdot F} (\text{ext}_\Psi f_2) \right) d\mu_h,$$

where $d\mu_h$ is the standard volume element of \mathcal{N} . Notice that the argument of the last integral is null at least outside $\psi(M)$ because of the definition of ext_Ψ . Moreover the restriction to $\psi(M)$ of the vector bundle F is the vector bundle $\Psi(E)$ (cfr. Remark 1.1.15), whose inner product is the restriction of the inner product of F (see few lines before eq. (2.2.2)). This gives us the opportunity to write

$$\omega(\xi u_1, \xi u_2) = \int_{\psi(M)} \left((e_B(\text{ext}_\Psi f_1))|_{\psi(M)}^{\Psi(E)} (\text{ext}_\Psi f_2)|_{\psi(M)} \right) d\mu_{h|_{\psi(M)}}.$$

Exploiting the definition of ext_Ψ , we find that

$$(\text{ext}_\Psi f_2)|_{\psi(M)} = \Psi' \circ f_2 \circ \psi'^{-1} \in \mathcal{D}(\psi(M), \Psi(E)).$$

Recalling that ψ is an isometric embedding, we also have $\psi'_* g = h|_{\psi(M)}$ and, as a consequence of the fact that (ψ'^{-1}, Ψ'^{-1}) is a morphism of \mathbf{ghs}^f from $(\psi(\mathcal{M}), \Psi(E), A_\Psi)$ to (\mathcal{M}, E, A) (we noted this fact few lines after eq. (2.2.3)), we deduce that

$$\begin{aligned} \omega(\xi u_1, \xi u_2) &= \int_{\psi(M)} \left((e_B(\text{ext}_\Psi f_1))|_{\psi(M)}^{\Psi(E)} (\Psi' \circ f_2 \circ \psi'^{-1}) \right) d\mu_{\psi'_* g} \\ &= \int_M \left(\left(\Psi'^{-1} \circ (e_B(\text{ext}_\Psi f_1))|_{\psi(M)} \circ \psi' \right)^{\cdot E} f_2 \right) d\mu_g \\ &= \int_M \left(((\text{res}_\Psi \circ e_B \circ \text{ext}_\Psi) f_1)^{\cdot E} f_2 \right) d\mu_g. \end{aligned}$$

We apply Lemma 2.2.4 and, recalling the definition of σ given in Lemma 2.2.3, we

conclude the proof:

$$\begin{aligned}\omega(\xi u_1, \xi u_2) &= \int_M \left(((\text{res}_\Psi \circ e_B \circ \text{ext}_\Psi) f_1)^E \cdot f_2 \right) d\mu_g = \int_M \left((e_A f_1)^E \cdot f_2 \right) d\mu_g \\ &= \sigma(u_1, u_2).\end{aligned}$$

□

Now we have the second part of our candidate covariant functor: for each pair of objects (\mathcal{M}, E, A) and (\mathcal{N}, F, B) of \mathbf{ghs}^f there exists a map

$$\begin{aligned}\mathcal{B} : \text{Mor}_{\mathbf{ghs}^f}((\mathcal{M}, E, A), (\mathcal{N}, F, B)) &\rightarrow \text{Mor}_{\mathbf{ssp}}(\mathcal{B}(\mathcal{M}, E, A), \mathcal{B}(\mathcal{N}, F, B)) \\ (\psi, \Psi) &\mapsto \xi\end{aligned}$$

defined in accordance with Lemma 2.2.5. To complete the theory of the classical field under consideration, it remains only to check that \mathcal{B} is actually a covariant functor. The next theorem answers to this question.

Theorem 2.2.6. *The map $\mathcal{B} : \text{Obj}_{\mathbf{ghs}^f} \rightarrow \text{Obj}_{\mathbf{ghs}^f}$ defined in accordance with Lemma 2.2.3, together with the collection of maps*

$$\left\{ \begin{array}{l} \mathcal{B} : \text{Mor}_{\mathbf{ghs}^f}((\mathcal{M}, E, A), (\mathcal{N}, F, B)) \rightarrow \text{Mor}_{\mathbf{ssp}}(\mathcal{B}(\mathcal{M}, E, A), \mathcal{B}(\mathcal{N}, F, B)) \\ \text{for } (\mathcal{M}, E, A), (\mathcal{N}, F, B) \in \text{Obj}_{\mathbf{ghs}^f} \end{array} \right\}$$

defined few lines above, gives rise to a covariant functor \mathcal{B} from the category \mathbf{ghs}^f to the category \mathbf{ssp} . Moreover \mathcal{B} possesses the following properties:

- *causality: for each $(\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t}), E, A)$, $(\mathcal{M}_1 = (M_1, g_1, \mathfrak{o}_1, \mathfrak{t}_1), E_1, A_1)$, $(\mathcal{M}_2 = (M_2, g_2, \mathfrak{o}_2, \mathfrak{t}_2), E_2, A_2)$ in $\text{Obj}_{\mathbf{ghs}^f}$, each morphism (ψ_1, Ψ_1) of \mathbf{ghs}^f from $(\mathcal{M}_1, E_1, A_1)$ to (\mathcal{M}, E, A) and each morphism (ψ_2, Ψ_2) from $(\mathcal{M}_2, E_2, A_2)$ to (\mathcal{M}, E, A) such that $\psi_1(M_1)$ and $\psi_2(M_2)$ are \mathcal{M} -causally separated subsets of M , it holds that*

$$\sigma(\xi_1 u_1, \xi_2 u_2) = 0$$

for each $u_1 \in V_1$ and each $u_2 \in V_2$, where (V, σ) , (V_1, σ_1) , (V_2, σ_2) are the symplectic spaces obtained applying \mathcal{B} respectively to (\mathcal{M}, E, A) , $(\mathcal{M}_1, E_1, A_1)$, $(\mathcal{M}_2, E_2, A_2)$ and ξ_1, ξ_2 are the symplectic maps obtained applying \mathcal{B} respectively to (ψ_1, Ψ_1) , (ψ_2, Ψ_2) ;

- *time slice axiom: for each $(\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t}), E, A)$, $(\mathcal{N} = (N, h, \mathfrak{p}, u), F, B)$ in $\text{Obj}_{\mathbf{ghs}^f}$ and each morphism (ψ, Ψ) of \mathbf{ghs}^f from (\mathcal{M}, E, A) to (\mathcal{N}, F, B) such that $\psi(M)$ contains a smooth spacelike Cauchy surface Σ for \mathcal{N} , it holds that*

$$\xi(V) = W,$$

where (V, σ) , (W, ω) are the symplectic spaces obtained applying \mathcal{B} respectively to (\mathcal{M}, E, A) , (\mathcal{N}, F, B) and ξ is the symplectic map obtained applying \mathcal{B} to (ψ, Ψ) . In particular ξ is bijective and its inverse ξ^{-1} is a morphism of \mathbf{ssp} from (W, ω) to (V, σ) .

Proof. We must check that \mathcal{B} satisfies the covariant axioms of Definition 1.5.3. Consider three objects (\mathcal{M}, E, A) , (\mathcal{N}, F, B) and (\mathcal{O}, G, C) of \mathbf{ghs}^f , a morphism (ϕ, Φ) from (\mathcal{M}, E, A) to (\mathcal{N}, F, B) and a morphism (ψ, Ψ) from (\mathcal{N}, F, B) to (\mathcal{O}, G, C) . Our aim is to show that the composition is preserved by \mathcal{B} , i.e.

$$\mathcal{B}((\psi, \Psi) \circ (\phi, \Phi)) = \mathcal{B}(\psi, \Psi) \circ \mathcal{B}(\phi, \Phi).$$

For each $u \in V$, where $V = \mathcal{B}(\mathcal{M}, E, A)$, we find $f \in \mathcal{D}(M, E)$ such that $e_A f = u$. This allows us to evaluate the LHS of our last equation:

$$\mathcal{B}((\psi, \Psi) \circ (\phi, \Phi)) u = e_C (\text{ext}_{\Psi \circ \Phi} f).$$

For the RHS we have

$$(\mathcal{B}(\psi, \Psi) \circ \mathcal{B}(\phi, \Phi)) u = \mathcal{B}(\psi, \Psi) (e_B (\text{ext}_{\Phi} f)) = e_C (\text{ext}_{\Psi} (\text{ext}_{\Phi} f)).$$

Recalling eq. (2.2.5) and comparing our last two equations, we deduce that

$$\mathcal{B}((\psi, \Psi) \circ (\phi, \Phi)) u = (\mathcal{B}(\psi, \Psi) \circ \mathcal{B}(\phi, \Phi)) u$$

for each $u \in V$, that is exactly what we wanted to prove. Now we consider the identity morphism $\text{id}_{(\mathcal{M}, E, A)}$ of $\mathbf{Mor}_{\mathbf{ghs}^f}((\mathcal{M}, E, A), (\mathcal{M}, E, A))$. We immediately realize that such morphism is provided by the identity maps of the sets M and E :

$$\text{id}_{(\mathcal{M}, E, A)} = (\text{id}_M, \text{id}_E).$$

The identity morphism $\text{id}_{(V, \sigma)} \in \mathbf{Mor}_{\mathbf{ssp}}((V, \sigma), (V, \sigma))$, for $(V, \sigma) = \mathcal{B}(\mathcal{M}, E, A)$, is provided by the identity map of the set V too:

$$\text{id}_{(V, \sigma)} = \text{id}_V.$$

We want to show that

$$\mathcal{B}(\text{id}_{(\mathcal{M}, E, A)}) = \text{id}_{(V, \sigma)}.$$

We consider $u \in V$ and, taking $f \in \mathcal{D}(M, E)$ such that $e_A f = u$, we obtain

$$\mathcal{B}(\text{id}_{(\mathcal{M}, E, A)}) u = e_A (\text{ext}_{\text{id}_E} f) = e_A (\text{id}_E \circ f \circ \text{id}_M^{-1}) = e_A f = u = \text{id}_{(V, \sigma)} u.$$

This equation holds for each $u \in V$. We deduce that \mathcal{B} maps the identity morphisms

of \mathbf{ghs}^f to the identity morphisms of \mathbf{ssp} . We have shown that \mathcal{B} is actually a covariant functor from \mathbf{ghs}^f to \mathbf{ssp} .

We turn our attention to the causality property. Let $(\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t}), E, A)$, $(\mathcal{M}_1 = (M_1, g_1, \mathfrak{o}_1, \mathfrak{t}_1), E_1, A_1)$ and $(\mathcal{M}_2 = (M_2, g_2, \mathfrak{o}_2, \mathfrak{t}_2), E_2, A_2)$ be objects of \mathbf{ghs}^f and suppose that (ψ_1, Ψ_1) is a morphism from $(\mathcal{M}_1, E_1, A_1)$ to (\mathcal{M}, E, A) and that (ψ_2, Ψ_2) is a morphism from $(\mathcal{M}_2, E_2, A_2)$ to (\mathcal{M}, E, A) . Moreover assume that $\psi_1(M_1)$ and $\psi_2(M_2)$ are \mathcal{M} -causally separated subsets of M . We denote with (V, σ) , (V_1, σ_1) and (V_2, σ_2) the symplectic spaces associated respectively to (\mathcal{M}, E, A) , $(\mathcal{M}_1, E_1, A_1)$ and $(\mathcal{M}_2, E_2, A_2)$ via \mathcal{B} and with ξ_1 and ξ_2 the symplectic maps associated respectively to (ψ_1, Ψ_1) and (ψ_2, Ψ_2) . Consider two elements $u_1 \in V_1$ and $u_2 \in V_2$. We surely find $f_1 \in \mathcal{D}(M_1, E_1)$ such that $e_{A_1} f_1 = u_1$ and $f_2 \in \mathcal{D}(M_2, E_2)$ such that $e_{A_2} f_2 = u_2$. This allows us to evaluate $\xi_1 u_1$ and $\xi_2 u_2$:

$$\begin{aligned}\xi_1 u_1 &= e_A(\text{ext}_{\Psi_1} f_1), \\ \xi_2 u_2 &= e_A(\text{ext}_{\Psi_2} f_2).\end{aligned}$$

Notice that $\text{supp}(f_1)$ is a compact subset of M_1 and therefore $\text{supp}(\text{ext}_{\Psi_1} f_1)$ is a compact subset of M included in $\psi_1(M_1)$. Similarly $\text{supp}(\text{ext}_{\Psi_2} f_2)$ is a compact subset of M included in $\psi_2(M_2)$. Due to the support property of the advanced and retarded Green operators for A (see Definition 1.3.15), we have that

$$\text{supp}(\xi_1 u_1) \subseteq J^{\mathcal{M}}(\text{supp}(\text{ext}_{\Psi_1} f_1)) \subseteq J^{\mathcal{M}}(\psi_1(M_1)).$$

We assumed that $\psi_1(M_1)$ and $\psi_2(M_2)$ are \mathcal{M} -causally separated subsets of M , therefore, via Remark 1.2.8, we obtain

$$\text{supp}(\xi_1 u_1) \cap \text{supp}(\xi_2 u_2) \subseteq J^{\mathcal{M}}(\psi_1(M_1)) \cap \psi_2(M_2) = \emptyset.$$

These observations give us the opportunity to evaluate $\sigma(\xi_1 u_1, \xi_2 u_2)$:

$$\sigma(\xi_1 u_1, \xi_2 u_2) = \int_M \left((\xi_1 u_1)^E \cdot (\text{ext}_{\Psi_2} f_2) \right) d\mu_g = 0$$

because the support of the integrand is empty. This shows that causality holds.

We are left only with the check of the time slice axiom. Consider two objects $(\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t}), E, A)$ and $(\mathcal{N} = (N, h, \mathfrak{p}, u), F, B)$ of \mathbf{ghs}^f and suppose that (ψ, Ψ) is a morphism from (\mathcal{M}, E, A) to (\mathcal{N}, F, B) whose image $\psi(M)$ includes a smooth spacelike Cauchy surface Σ for \mathcal{N} . We denote with (V, σ) and (W, ω) the symplectic spaces obtained through \mathcal{B} from (\mathcal{M}, E, A) and respectively (\mathcal{N}, F, B) and we impose $\xi = \mathcal{B}(\psi, \Psi)$. W is codomain of ξ , hence the inclusion $\xi(V) \subseteq W$ is trivial and we must prove the converse inclusion to complete the proof. To this end consider $u \in W$. We look for a section $f \in \mathcal{D}(M, E)$ such that $e_B(\text{ext}_{\Psi} f) = u$.

We observe that u is obtained from a compactly supported section in F through the causal propagator e_B . Hence, exploiting the support properties of the Green operators, we find a compact subset K of N such that

$$\text{supp}(u) \subseteq J^{\mathcal{N}}(K).$$

The problem is that, in general, K is not included in $\psi(M)$. Anyway we can go around this obstacle with the following procedure. We note that $\text{supp}(u) \cap \Sigma$ is a compact subset of Σ (intended as a topological space in its own right with the topology induced by the topology of N) because of Proposition 1.2.18. Take a \mathbf{u} -future directed h -timelike unit vector field \mathbf{n} over Σ normal to Σ (such vector field actually exists because Σ is spacelike). Considering Σ as a $(d-1)$ -dimensional submanifold of N and introducing the vector bundle $F|_{\Sigma} = \pi_F^{-1}(\Sigma)$, we can define two compactly supported sections over Σ :

$$\begin{aligned} u_0 : \Sigma &\rightarrow F|_{\Sigma} & \text{and} & & u_1 : \Sigma &\rightarrow F|_{\Sigma} \\ q &\mapsto u(q) & & & q &\mapsto (\nabla_{\mathbf{n}} u)(q), \end{aligned}$$

where ∇ is the B -compatible connection in F . Since u_0 and u_1 fall in $\mathcal{D}(\Sigma, E|_{\Sigma})$, we can use them to formulate a well posed Cauchy problem for the normally hyperbolic operator B :

$$\begin{cases} Bv &= 0, \\ v|_{\Sigma} &= u_0, \\ \nabla_{\mathbf{n}} v|_{\Sigma} &= u_1. \end{cases}$$

Theorem 1.3.7 tells us that the Cauchy problem stated above admits exactly one solution $v \in C^{\infty}(N, F)$ whose support is contained in

$$J^{\mathcal{N}}(\text{supp}(u_0) \cup \text{supp}(u_1)) \subseteq J^{\mathcal{N}}(\text{supp}(u) \cap \Sigma).$$

By construction u satisfies that Cauchy problem and therefore $u = v$ for uniqueness, in particular u and v have the same support. Since $\text{supp}(u) \cap \Sigma$ is a compact subset of $\Sigma \subseteq \psi(M)$, it is also a compact subset of N included in $\psi(M)$. This fact gives us the chance to find a compact subset K of N that contains $\text{supp}(u) \cap \Sigma$ and that is included in $\psi(M)$. Since K is compact, we can also find a relatively compact open subset Ω of N such that $K \subseteq \Omega \subseteq \psi(M)$. Using Ω , we can introduce a covering of N :

$$\{J_+^{\mathcal{N}}(\Omega), J_-^{\mathcal{N}}(\Omega), N \setminus J^{\mathcal{N}}(K)\}.$$

This is an open covering because $J_{\pm}^{\mathcal{N}}(\Omega)$ are open subsets of N (see [FV11, Lem. A.8, p. 48]) and $J_{\pm}^{\mathcal{N}}(K)$ are closed subsets of N (see [BGP07, Lem. A.5.1, p. 173]).

Then we can introduce a partition of unity subordinate to such covering:

$$\{\chi^+, \chi^-, \chi^0\}.$$

Then we define $u^\pm = \chi^\pm u$ and $u^0 = \chi^0 u$ and we have $u = u^+ + u^- + u^0$. As a consequence of our construction

$$\begin{aligned} \text{supp}(u^0) &= \text{supp}(\chi^0) \cap \text{supp}(u) \subseteq (N \setminus J^{\mathcal{N}}(K)) \cap J^{\mathcal{N}}(\text{supp}(u) \cap \Sigma) \\ &\subseteq (N \setminus J^{\mathcal{N}}(K)) \cap J^{\mathcal{N}}(K) = \emptyset \end{aligned}$$

and therefore u^0 is everywhere null. This implies that $u = u^+ + u^-$. Since we know that $Bu = 0$, we deduce that $Bu^+ = -Bu^-$. In particular this relation implies that

$$\text{supp}(Bu^+) \subseteq \text{supp}(\chi^+) \cap \text{supp}(\chi^-) \subseteq J_+^{\mathcal{N}}(\Omega) \cap J_-^{\mathcal{N}}(\Omega) \subseteq J_+^{\mathcal{N}}(\overline{\Omega}) \cap J_-^{\mathcal{N}}(\overline{\Omega}),$$

where $\overline{\Omega}$ denotes the closure of Ω in N . Since Ω is relatively compact in N , $\overline{\Omega}$ is compact in N . Applying Proposition 1.2.18, we deduce that $J_+^{\mathcal{N}}(\overline{\Omega}) \cap J_-^{\mathcal{N}}(\overline{\Omega})$ is a compact subset of N and therefore $Bu^+ = -Bu^-$ is a section in F with compact support. We are able to find more information about its support:

$$\text{supp}(Bu^+) \subseteq J_+^{\mathcal{N}}(\Omega) \cap J_-^{\mathcal{N}}(\Omega) \subseteq \psi(M).$$

We prove this inclusion: Consider $p \in J_+^{\mathcal{N}}(\Omega) \cap J_-^{\mathcal{N}}(\Omega)$; we find a \mathbf{u} -future directed h -causal curve γ_1 in N from $q \in \Omega$ to p and a \mathbf{u} -past directed h -causal curve γ_2 in N from $r \in \Omega$ to p ; reversing the direction of γ_2 and pasting the result with γ_1 , we obtain a \mathbf{u} -future directed h -causal curve γ in N from q to r ; both q and r fall in $\psi(M)$ because $\Omega \subseteq \psi(M)$ by construction; since $\psi(M)$ is \mathcal{N} -causally convex by hypothesis, γ must be entirely contained in $\psi(M)$, in particular $p \in \psi(M)$. At this point we have a section $Bu^+ \in \mathcal{D}(N, F)$ with support included in $\psi(M)$. We use it to define a compactly supported section in E via restriction:

$$f = \text{res}_\Psi(Bu^+) = -\text{res}_\Psi(Bu^-) \in \mathcal{D}(M, E).$$

Now we check that f is exactly the one we were looking for. First of all f has compact support so that $\text{ext}_\Psi f$ has compact support too, hence we can apply $e_B^{a/r}$ to it and we obtain

$$\begin{aligned} e_B^a(\text{ext}_\Psi f) &= e_B^a(Bu^+), \\ e_B^r(\text{ext}_\Psi f) &= -e_B^r(Bu^-). \end{aligned}$$

Now we observe that

$$\begin{aligned}\operatorname{supp}(u^+) &\subseteq J_+^{\mathcal{N}}(\overline{\Omega}), \\ \operatorname{supp}(u^-) &\subseteq J_-^{\mathcal{N}}(\overline{\Omega})\end{aligned}$$

and Proposition 1.2.18 implies that u^+ has \mathcal{N} -past compact support, while u^- has \mathcal{N} -future compact support. This fact allows us to apply Lemma 1.3.17 to obtain

$$\begin{aligned}e_B^a(\operatorname{ext}_{\Psi}f) &= u^+, \\ e_B^r(\operatorname{ext}_{\Psi}f) &= -u^-\end{aligned}$$

and therefore

$$e_B(\operatorname{ext}_{\Psi}f) = u^+ - (-u^-) = u.$$

This completes our proof because, setting $w = e_A f \in V$, we have $\xi w = e_B(\operatorname{ext}_{\Psi}f) = u$, hence in particular $u \in \xi(V)$ and this fact, for the freedom in the choice of $u \in W$, implies the inclusion $W \subseteq \xi(V)$. The last part of the statement of the time slice axiom follows directly because each symplectic map is automatically injective (cfr. Remark 1.4.10) and the time slice axiom assures that ξ is also surjective, hence the inverse ξ^{-1} exists and it is trivial to check that it is a symplectic map too. \square

2.2.2 ...to quantum field theory

In the last subsection we built the theory of a classical field over some d -dimensional globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ modeled by a smooth section u in a vector bundle E of rank n over M satisfying the normally hyperbolic equation $Au = 0$ on each point of M , where E is endowed with an inner product denoted by $\overset{E}{\cdot}$ and A is a formally selfadjoint normally hyperbolic operator on E over \mathcal{M} . Now we want to use this result to build the quantum field theory that corresponds to this situation. Most of the work has already been done in the previous subsection and in Subsection 1.4.1. Here we simply put the pieces of the puzzle together. We start building a new covariant functor from the category **ssp** to the category **alg** and then we compose it with \mathcal{B} . As we will see, this will give us a covariant functor \mathcal{A} that is actually a locally covariant quantum field theory fulfilling both the causality condition and the time slice axiom. As a consequence of our Theorem 2.1.9, on each globally hyperbolic spacetime \mathcal{M} the LCQFT \mathcal{A} provides the quantum field theory (in the sense of the Haag-Kastler approach) of the field under consideration.

Lemma 2.2.7. *Consider a map \mathcal{C} that associates to each symplectic space (V, σ) its unique (up to $*$ -isomorphisms) CCR representation $(\mathcal{V}, \mathbb{V})$ in accordance with Definition 1.4.13 and for each pair of symplectic spaces (V, σ) and (W, ω) , whose corresponding CCR representations are respectively $(\mathcal{V}, \mathbb{V}) = \mathcal{C}(V, \sigma)$ and $(\mathcal{W}, \mathbb{W}) = \mathcal{C}(W, \omega)$, consider a map \mathcal{C} that associates to each symplectic map ξ from (V, σ)*

to (W, ω) the unique injective unit preserving $*$ -homomorphism H from (\mathcal{V}, V) to (\mathcal{W}, W) in accordance with Proposition 1.4.16 and the subsequent observation. Then \mathcal{C} is a covariant functor from the category \mathbf{ssp} to the category \mathbf{alg} .

Proof. Consider a symplectic space (V, σ) . Shortly after Definition 1.4.11, we observed that there exists at least one Weyl system associated to each symplectic space. We consider the unital sub-C*-algebra \mathcal{V} of the Weyl algebra under consideration generated by the image of the Weyl map. This gives rise to a CCR representation (\mathcal{V}, V) of (V, σ) as it can be directly checked via Definition 1.4.13. (\mathcal{V}, V) is unique up to $*$ -isomorphism as a consequence of Proposition 1.4.14. Then a map of the type required in the statement is obtained imposing $\mathcal{C}(V, \sigma) = (\mathcal{V}, V)$. Note that we have just defined \mathcal{C} as a map from $\mathbf{Obj}_{\mathbf{ssp}}$ to $\mathbf{Obj}_{\mathbf{alg}}$.

Consider now a pair of symplectic spaces (V, σ) and (W, ω) , let (\mathcal{V}, V) and (\mathcal{W}, W) denote respectively $\mathcal{C}(V, \sigma)$ and $\mathcal{C}(W, \omega)$ and suppose that ξ is a symplectic map from (V, σ) to (W, ω) . Applying Proposition 1.4.16 and the subsequent observation, we obtain a unique injective unit preserving $*$ -homomorphism $H : \mathcal{V} \rightarrow \mathcal{W}$ satisfying $H \circ V = W \circ \xi$. Then we define a map \mathcal{C} as required by the statement setting $\mathcal{C}(\xi) = H$. Note that we have just defined \mathcal{C} as a map from $\mathbf{Mor}_{\mathbf{ssp}}((V, \sigma), (W, \omega))$ to $\mathbf{Mor}_{\mathbf{alg}}((\mathcal{V}, V), (\mathcal{W}, W))$.

At this point \mathcal{C} is a good candidate to become a covariant functor from \mathbf{ssp} to \mathbf{alg} , but we have still to check the covariant axioms. We begin checking that \mathcal{C} preserves the composition. To this end we consider three objects (U, ρ) , (V, σ) and (W, ω) of \mathbf{ssp} and we denote the images of these objects through \mathcal{C} respectively with (\mathcal{U}, U) , (\mathcal{V}, V) and (\mathcal{W}, W) . Moreover we take a morphism ξ of \mathbf{ssp} from (U, ρ) to (V, σ) and a morphism η of \mathbf{ssp} from (V, σ) to (W, ω) . $\eta \circ \xi$ is undoubtedly a morphism of \mathbf{ssp} from (U, ρ) to (W, ω) . Then we can consider the following morphisms of \mathbf{alg} :

$$\begin{aligned}\mathcal{C}(\xi) &\in \mathbf{Mor}_{\mathbf{alg}}((\mathcal{U}, U), (\mathcal{V}, V)), \\ \mathcal{C}(\eta) &\in \mathbf{Mor}_{\mathbf{alg}}((\mathcal{V}, V), (\mathcal{W}, W)), \\ \mathcal{C}(\eta \circ \xi) &\in \mathbf{Mor}_{\mathbf{alg}}((\mathcal{U}, U), (\mathcal{W}, W)).\end{aligned}$$

We also have that these morphisms satisfy the following relations:

$$\begin{aligned}\mathcal{C}(\xi) \circ U &= V \circ \xi, \\ \mathcal{C}(\eta) \circ V &= W \circ \eta, \\ \mathcal{C}(\eta \circ \xi) \circ U &= W \circ (\eta \circ \xi).\end{aligned}$$

Surely $\mathcal{C}(\eta) \circ \mathcal{C}(\xi)$ is a morphism of \mathbf{alg} from (\mathcal{U}, U) to (\mathcal{W}, W) such as $\mathcal{C}(\eta \circ \xi)$ and, exploiting the first two equations, we get

$$\mathcal{C}(\eta) \circ \mathcal{C}(\xi) \circ U = \mathcal{C}(\eta) \circ V \circ \xi = W \circ \eta \circ \xi.$$

Proposition 1.4.16 tells us that there exists a unique injective $*$ -homomorphism H from (\mathcal{U}, U) to (\mathcal{W}, W) such that $H \circ W = U \circ (\eta \circ \xi)$, hence $\mathcal{C}(\eta \circ \xi) = \mathcal{C}(\eta) \circ \mathcal{C}(\xi)$ and then \mathcal{C} preserves the composition of morphisms. To conclude we check that \mathcal{C} maps the identity morphisms to the identity morphisms. To this end we consider the object (V, σ) of \mathbf{ssp} and its image (\mathcal{V}, V) through \mathcal{C} . It is easy to check that the identity morphism $\text{id}_{(V, \sigma)}$ of (V, σ) is provided by the identity map id_V of the set V and that the identity morphism $\text{id}_{(\mathcal{V}, V)}$ of (\mathcal{V}, V) is provided by the identity map $\text{id}_{\mathcal{V}}$ of the set \mathcal{V} . Together with $\text{id}_{(\mathcal{V}, V)}$, we can consider another morphism of \mathbf{alg} from (\mathcal{V}, V) to itself, specifically $\mathcal{C}(\text{id}_{(V, \sigma)})$. On the one hand we have

$$\text{id}_{(\mathcal{V}, V)}(V(v)) = V(v) = V(\text{id}_{(V, \sigma)}v) \quad \forall v \in V,$$

which means exactly

$$\text{id}_{(\mathcal{V}, V)} \circ V = V \circ \text{id}_{(V, \sigma)},$$

while on the other side, exploiting the definition of \mathcal{C} , we obtain

$$\mathcal{C}(\text{id}_{(V, \sigma)}) \circ V = V \circ \text{id}_{(V, \sigma)}.$$

Applying Proposition 1.4.16 as we did above, we find that

$$\mathcal{C}(\text{id}_{(V, \sigma)}) = \text{id}_{(\mathcal{V}, V)},$$

which is to say that \mathcal{C} maps identity morphisms to identity morphisms. This completes the proof. \square

At this point we have the covariant functors $\mathcal{B} : \mathbf{ghs}^f \rightarrow \mathbf{ssp}$ and $\mathcal{C} : \mathbf{ssp} \rightarrow \mathbf{alg}$ and we can compose them in accordance with Definition 1.5.5 to obtain a new covariant functor. We present the result in the next theorem.

Theorem 2.2.8. *Consider the covariant functor $\mathcal{B} : \mathbf{ghs}^f \rightarrow \mathbf{ssp}$ defined in Theorem 2.2.6 and the covariant functor $\mathcal{C} : \mathbf{ssp} \rightarrow \mathbf{alg}$ defined in Lemma 2.2.7. Then $\mathcal{A} = \mathcal{C} \circ \mathcal{B}$ is a locally covariant quantum field theory that fulfils the causality condition and the time slice axiom.*

Proof. The composition of covariant functors yields a covariant functor (see Definition 1.5.5), hence $\mathcal{A} = \mathcal{C} \circ \mathcal{B}$ is a covariant functor from the category \mathbf{ghs}^f to the category \mathbf{alg} . Besides the richer content of the category \mathbf{ghs}^f compared to \mathbf{ghs} (recall the discussion at the beginning of this section), nonetheless we recognize \mathcal{A} to be a LCQFT (cfr. Definition 2.1.5) in light of the discussion at the beginning of this section at page 95.

Now we check that \mathcal{A} fulfils the causality condition of Definition 2.1.5. To this end consider three objects $(\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t}), E, A)$, $(\mathcal{M}_1 = (M_1, g_1, \mathfrak{o}_1, \mathfrak{t}_1), E_1, A_1)$ and $(\mathcal{M}_2 = (M_2, g_2, \mathfrak{o}_2, \mathfrak{t}_2), E_2, A_2)$ in \mathbf{ghs}^f , a morphism (ψ_1, Ψ_1) from $(\mathcal{M}_1, E_1, A_1)$

to (\mathcal{M}, E, A) and a morphism (ψ_2, Ψ_2) from $(\mathcal{M}_2, E_2, A_2)$ to (\mathcal{M}, E, A) and suppose that $\psi_1(M_1)$ and $\psi_2(M_2)$ are \mathcal{M} -causally separated subsets of M . Denote the symplectic spaces $\mathcal{B}(\mathcal{M}, E, A)$, $\mathcal{B}(\mathcal{M}_1, E_1, A_1)$ and $\mathcal{B}(\mathcal{M}_2, E_2, A_2)$ respectively with (V, σ) , (V_1, σ_1) and (V_2, σ_2) and the symplectic maps $\mathcal{B}(\psi_1, \Psi_1)$ and $\mathcal{B}(\psi_2, \Psi_2)$ respectively with ξ_1 and ξ_2 . Moreover denote the CCR representations $\mathcal{A}(\mathcal{M}, E, A) = \mathcal{C}(V, \sigma)$, $\mathcal{A}(\mathcal{M}_1, E_1, A_1) = \mathcal{C}(V_1, \sigma_1)$ and $\mathcal{A}(\mathcal{M}_2, E_2, A_2) = \mathcal{C}(V_2, \sigma_2)$ respectively with $(\mathcal{V}, \mathbf{V})$, $(\mathcal{V}_1, \mathbf{V}_1)$ and $(\mathcal{V}_2, \mathbf{V}_2)$ and the injective unit preserving $*$ -homomorphisms $\mathcal{A}(\psi_1, \Psi_1) = \mathcal{C}(\xi_1)$ and $\mathcal{A}(\psi_2, \Psi_2) = \mathcal{C}(\xi_2)$ respectively with H_1 and H_2 . Theorem 2.2.6 tells us that \mathcal{B} satisfies the causality property, i.e.

$$\sigma(\xi_1 u_1, \xi_2 u_2) = 0 \quad (2.2.6)$$

for each $u_1 \in V_1$ and each $u_2 \in V_2$. We want to show that

$$[H_1(V_1(u_1)), H_2(V_2(u_2))] = 0.$$

Exploiting the definitions of H_1 and H_2 (cfr. Lemma 2.2.7), we find

$$\begin{aligned} H_1(V_1(u_1)) &= \mathbf{V}(\xi_1 u_1), \\ H_2(V_2(u_2)) &= \mathbf{V}(\xi_2 u_2). \end{aligned}$$

This fact, together with the properties of the Weyl map \mathbf{V} (cfr. Definition 1.4.11) and eq. (2.2.6), allows us to evaluate the commutator above:

$$\begin{aligned} [H_1(V_1(u_1)), H_2(V_2(u_2))] &= [\mathbf{V}(\xi_1(u_1)), \mathbf{V}(\xi_2(u_2))] \\ &= \mathbf{V}(\xi_1 u_1) \mathbf{V}(\xi_2 u_2) - \mathbf{V}(\xi_2 u_2) \mathbf{V}(\xi_1 u_1) \\ &= (e^{-\frac{i}{2}\sigma(\xi_1 u_1, \xi_2 u_2)} - e^{-\frac{i}{2}\sigma(\xi_2 u_2, \xi_1 u_1)}) \mathbf{V}(\xi_1 u_1 + \xi_2 u_2) \\ &= 0. \end{aligned}$$

The last relation implies that

$$[H_1(a_1), H_2(a_2)] = 0$$

for each $a_1 \in \mathcal{V}_1$ and each $a_2 \in \mathcal{V}_2$ because $\mathbf{V}_1(V_1)$ is the set of generators of \mathcal{V}_1 , $\mathbf{V}_2(V_2)$ is the set of generators of \mathcal{V}_2 (by Definition of CCR representation), both H_1 and H_2 are continuous (cfr. Proposition 1.4.7) and also the sum and the multiplication of \mathcal{V} are continuous. This means that \mathcal{A} fulfils the causality condition as we stated it in Definition 2.1.5.

As for the time slice axiom, consider two objects $(\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t}), E, A)$ and $(\mathcal{N} = (N, h, \mathfrak{p}, \mathfrak{u}), F, B)$ of \mathbf{ghs}^f and a morphism (ψ, Ψ) between (\mathcal{M}, E, A) and (\mathcal{N}, F, B) such that $\psi(M)$ contains a smooth spacelike Cauchy surface Σ for \mathcal{N} . We denote the symplectic spaces $\mathcal{B}(\mathcal{M}, E, A)$ and $\mathcal{B}(\mathcal{N}, F, B)$ respectively with

(V, σ) and (W, ω) and the symplectic map $\mathcal{B}(\psi, \Psi)$ with ξ . Moreover we denote the CCR representations $\mathcal{A}(\mathcal{M}, E, A) = \mathcal{C}(V, \sigma)$ and $\mathcal{A}(\mathcal{N}, F, B) = \mathcal{C}(W, \omega)$ respectively with (\mathcal{V}, V) and (\mathcal{W}, W) and the injective unit preserving $*$ -homomorphism $\mathcal{A}(\psi, \Psi) = \mathcal{C}(\xi)$ with H . From Theorem 2.2.6 we know that \mathcal{B} satisfies the version of the time slice axiom for covariant functors from \mathbf{ghs}^f to \mathbf{alg} , i.e. $\xi(V) = W$, and our aim is to show that H is surjective, which is to say that \mathcal{A} satisfies the time slice axiom as a LCQFT. As we noted in Theorem 2.2.6, in the present situation ξ is bijective and its inverse ξ^{-1} is a symplectic map from (W, ω) to (V, σ) . Via the functor \mathcal{C} we obtain the injective unit preserving $*$ -homomorphism $\mathcal{C}(\xi^{-1})$ and then the covariant axioms imply that

$$\begin{aligned} H \circ \mathcal{C}(\xi^{-1}) &= \mathcal{C}(\xi) \circ \mathcal{C}(\xi^{-1}) = \text{id}_{(W, \omega)}, \\ \mathcal{C}(\xi^{-1}) \circ H &= \mathcal{C}(\xi^{-1}) \circ \mathcal{C}(\xi) = \text{id}_{(V, \sigma)}. \end{aligned}$$

This means that H is bijective and its inverse is $H^{-1} = \mathcal{C}(\xi^{-1})$. In particular H is surjective, as we wanted to show. \square

Remark 2.2.9. Since we recognized \mathcal{A} to be a LCQFT, we are allowed to apply Theorem 2.1.9. This gives us the opportunity to recover the Haag-Kastler framework for the description of the quantum theory of the field we are dealing with. Therefore the functor \mathcal{A} gives actually a quantum field theory (in its axiomatic definition by Haag and Kastler) for the field under consideration on each globally hyperbolic spacetime. At this point however this conclusion is not true at all because, as we noted in Remark 2.1.10, we have not yet shown that on each globally hyperbolic spacetime the unital C^* -algebra obtained through a LCQFT is primitive, i.e. it admits a faithful irreducible representation on a Hilbert space. Anyway we can see that this property holds for the LCQFT \mathcal{A} that we have built right now. Actually this is a property of our functor \mathcal{C} because it maps objects of \mathbf{ssp} to CCR representations and each CCR algebra is primitive: Each unital C^* -algebra admits an irreducible representation π on a Hilbert space \mathcal{H} (cfr. [BR02, Lem. 2.3.23, p. 59]) and π is indeed a unit preserving $*$ -homomorphism from the unital C^* -algebra to the unital C^* -algebra of bounded operators on \mathcal{H} ; in our case the unital C^* -algebra is also a CCR representation, therefore, applying Proposition 1.4.15, we see that π must be injective too, i.e. faithful, and hence we have just found a faithful irreducible representation π on a Hilbert space for each CCR representation.

2.3 Examples

In the last section we have shown how to build a locally covariant quantum field theory that describes a field over an arbitrary globally hyperbolic spacetime which is ruled at a classical level by a wave equation on that spacetime written in terms of

a normally hyperbolic operator acting on sections in a proper vector bundle. More precisely, Subsection 2.2.1 was devoted to the construction of the field theory at a classical level consisting of a covariant functor that provides the solutions to all homogeneous Cauchy problems with compactly supported initial data on a given globally hyperbolic spacetime. We also established a causality property and a form of time slice axiom for such functor (cfr. Theorem 2.2.6). After that, in Subsection 2.2.2 we introduced a covariant functor that maps each symplectic space to a unital C^* -algebras, actually the unique (up to $*$ -isomorphisms) CCR representation of that symplectic space. We may regard this covariant functor as a “quantization” functor because, when composed with the covariant functor describing the classical theory, it gives rise to a LCQFT which is causal and fulfils the time slice axiom (in the sense of Definition 2.1.5). Theorem 2.1.9 and Remark 2.2.9 recover the Haag-Kastler axioms and in this way they assure that this LCQFT actually provides the quantum field theory (in its axiomatic definition made by Haag and Kastler) for the field under consideration on each globally hyperbolic spacetime.

In this section we want to show some realizations of LCQFTs in situations of physical interest, specifically we discuss the real Klein-Gordon field, the real Proca field and the electromagnetic field. We will discuss these fields in terms of k -forms. This is the typical approach for the Maxwell equations, but it is quite unusual to treat in this way the Klein-Gordon equation and the Proca equation. Anyway we will see that the more familiar equations are equivalent to those written in terms of k -forms.

We want to show from now that the d’Alembertian operator

$$\square_k = d\delta + \delta d : \Omega^k M \rightarrow \Omega^k M,$$

defined in terms of the exterior derivative d (see Proposition 1.1.34) and the codifferential δ (see Definition 1.1.41), is a formally selfadjoint normally hyperbolic operator for each k .

Proposition 2.3.1. *Let (M, g) be an orientable Lorentzian d -dimensional manifold and let \mathfrak{o} be a choice of the orientation. Then for each k the d’Alembertian operator \square_k defined above is a formally selfadjoint normally hyperbolic operator on $\Lambda^k M$ over M . Moreover the following identities hold on $\Omega^k M$:*

$$\begin{aligned}\square_{k+1}d &= d\square_k, \\ \square_{k-1}\delta &= \delta\square_k.\end{aligned}$$

Proof. $\Lambda^k M$ reduces to $M \times \{0\}$ for $k > d$ (see the comments immediately after Definition 1.1.30) so that the statement of the proposition becomes trivial. Then, without loss of generality, we can fix $k \in \{0, \dots, d\}$.

In first place we show that \square_k is a linear differential operator from $\Lambda^k M$ to

itself. To this end we fix a section $\mu \in \Omega^k M$ and a point $p \in M$ and we choose a coordinate neighborhood (U, V, ϕ) for p in M . On V we put the orientation $\phi_*(\mathfrak{o}|_U)$ and on $TV = V \times \mathbb{R}^d$ we set the inner product $\phi_*(g|_U)$. Then we choose an oriented orthonormal basis $\{e_1, \dots, e_d\}$ of $\Lambda_{\phi(p)}^1 V$ and we define the local 1-forms $dx^i \in \Lambda^1 V$ through the formula $dx^i = e_i$ on each point of V : $\{dx^1, \dots, dx^d\}$ is a basis of $\Lambda^1 V$. Now we are ready to express $\square_k \mu$ in local coordinates applying Proposition 1.1.34 and the comments just after Definition 1.1.41:

$$\phi_*(\square_k \mu|_U) = (d\delta + \delta d)(\phi_*(\mu|_U)).$$

We rewrite $\phi_*(\mu|_U)$ using the base of $\Lambda^k V$:

$$(\phi_*(\mu|_U))(x) = \frac{1}{k!} f_{i_1 \dots i_k}(x) dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

The result of the calculation gives a k -form over V whose coefficients in the basis $\{dx^{i_1} \wedge \dots \wedge dx^{i_k}\}$ consist of (very long) linear combinations of partial derivatives of $f_{i_1 \dots i_k}$ up to the second order with coefficients involving the metric (and its first order partial derivatives) and the Levi-Civita symbol. This is sufficient to understand that \square_k is actually a linear differential operator of second order. To show that it is also normally hyperbolic we report the final expression of the term involving second order partial derivatives of $f_{i_1 \dots i_k}$:

$$-g^{lm}(x) \frac{\partial^2 f_{i_1 \dots i_k}}{\partial x^l \partial x^m}(x) dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

where $(g^{ij}(x))$ is the inverse of the matrix $(g_{ij}(x))$ whose coefficients are defined by

$$g_{ij}(x) = g_{\phi^{-1}(x)}(\phi^*(dx^i), \phi^*(dx^j)).$$

From this we deduce that the principal symbol σ_{\square_k} of \square_k is the map

$$\begin{aligned} T^*M &\rightarrow \text{Hom}(\Lambda^k M, \Lambda^k M) \\ (p, \omega) &\mapsto -g^{lm}(\phi(p))(\phi_*\omega)_l(\phi_*\omega)_m \text{id}_{\Lambda^k M}, \end{aligned}$$

where $\phi_*\omega = (\phi_*\omega)_i dx^i \in T_{\phi(p)}^* V$. Noting that

$$g^{lm}(\phi(p))(\phi_*\omega)_l(\phi_*\omega)_m = g_p(\omega^\sharp, \omega^\sharp),$$

we conclude that \square_k is normally hyperbolic.

Formal selfadjointness is deduced from Proposition 1.1.47 using the non degenerate inner product on $\Omega^k M$ defined in Proposition 1.1.46. For each $\mu, \nu \in \Omega_0^k M$,

we have

$$\begin{aligned}
(\square_k \mu, \nu)_{g,k} &= (d\delta\mu, \nu)_{g,k} + (\delta d\mu, \nu)_{g,k} \\
&= (\delta\mu, \delta\nu)_{g,k-1} + (d\mu, d\nu)_{g,k+1} \\
&= (\delta d\mu, \nu)_{g,k} + (\mu, d\delta\nu)_{g,k} \\
&= (\mu, \square_k \nu)_{g,k}
\end{aligned}$$

and this means exactly that \square_k is formally selfadjoint.

The stated identities follow from $d^2 = 0$ and $\delta^2 = 0$:

$$\begin{aligned}
\square_{k+1} d &= (d\delta + \delta d) d = d\delta d = d(d\delta + \delta d) = d\square_k, \\
\square_{k-1} \delta &= (d\delta + \delta d) \delta = \delta d\delta = \delta(d\delta + \delta d) = \delta\square_k.
\end{aligned}$$

□

2.3.1 The Klein-Gordon field

This is the easiest of our examples because, as we will see, we can apply completely the procedure of Section 2.2.

We fix a value of the mass $m \geq 0$. The Klein-Gordon field of mass m on a 4-dimensional globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ is described by a section φ in the trivial tensor bundle $\Lambda^0 M = M \times \mathbb{R}$ (sometimes this bundle is called line bundle) that satisfies the equation

$$A\varphi = \square_0 \varphi + m^2 \varphi = 0. \quad (2.3.1)$$

The inner product of $\Lambda^0 M$ is provided by multiplication of real numbers on each fiber. The operator $m^2 \text{id}_{\Lambda^0 M}$ is trivially formally selfadjoint, hence Proposition 2.3.1 implies that A is a formally selfadjoint normally hyperbolic operator. These considerations allow us to recognize $(\mathcal{M}, \Lambda^0 M, A)$ as an object of \mathbf{ghs}^f . Consider a morphism (ψ, Ψ) of \mathbf{ghs}^f from $(\mathcal{M}, \Lambda^0 M, A)$ to another object $(\mathcal{N}, \Lambda^0 N, B)$. In this case the situation is considerably simplified if compared to the general case of a morphism of \mathbf{ghs}^f because now each fiber is nothing but the real line and hence the fact that Ψ must be fiberwise an isometric isomorphism of \mathbb{R} to itself, together with the condition of compatibility with A and B (that are always of the form $\square_0 + m^2 \text{id}_{\Lambda^0 M}$), implies that $\Psi = \text{id}_{\mathbb{R}}$.

Then, when we want to describe the Klein-Gordon field, we restrict our category \mathbf{ghs}^f to a category \mathbf{ghs}^{KG} whose objects are 4-dimensional globally hyperbolic spacetimes $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ with the line bundle $\Lambda^0 M$ as vector bundle on which we set the inner product induced by fiberwise multiplication of real numbers and the formally selfadjoint normally hyperbolic operator $\square_0 + m^2 \text{id}_{\Lambda^0 M}$, which is com-

pletely determined by the metric and the orientation of \mathcal{M} . The morphisms that we consider are the morphisms of \mathbf{ghs}^f between the objects of \mathbf{ghs}^{KG} . This entails that \mathbf{ghs}^{KG} is a full subcategory of \mathbf{ghs}^f .

Due to the conditions of compatibility with the inner products and the normally hyperbolic operators, a morphism of \mathbf{ghs}^{KG} reduces to a map from $\Lambda^0 M = M \times \mathbb{R}$ to $\Lambda^0 N = N \times \mathbb{R}$ that acts on each fiber as the identity: $\Psi_p(p, \mu) = (\psi(p), \mu)$ for each $p \in M$ and each $\mu \in \mathbb{R}$. Hence for each $u \in \Omega^0 M$ the extension of u through (ψ, Ψ) is nothing but an “extended” push-forward through ψ' of u (refer to eq. (2.2.1)):

$$\text{ext}_\Psi u = \begin{cases} (u \circ \psi'^{-1})(q) & \text{if } q \in \psi(M), \\ 0 & \text{if } q \in N \setminus \psi(M) \end{cases} = \text{ext}_{\iota_{\Lambda^0 \psi(M)}^{\Lambda^0 N}} (\psi'_* u).$$

With these considerations we realize that the Klein-Gordon field is simply a special case of our general discussion so that we can apply the procedure of Section 2.2 obtaining first the covariant functor describing the classical theory and then the quantization functor. By composition of these covariant functors we obtain a locally covariant quantum field theory for the Klein-Gordon field and Theorem 2.1.9 (see Remark 2.2.9 for primitivity) assures that this LCQFT provides on each globally hyperbolic spacetime \mathcal{M} a unital C*-algebra satisfying the Haag-Kastler axioms, hence it is actually the quantum field theory of the Klein-Gordon field on \mathcal{M} .

Remark 2.3.2. Our conclusion rely upon the assumption that the description we gave of the Klein-Gordon field in terms of a 0-form φ over a globally hyperbolic spacetime \mathcal{M} satisfying the equation $\square_0 \varphi + m^2 \varphi = 0$ is equivalent to the usual formulation consisting of a real valued smooth function φ over \mathcal{M} ruled by the equation

$$-\nabla^i \nabla_i \varphi + m^2 \varphi = 0, \quad (2.3.2)$$

where ∇ is the Levi-Civita connection, $(g^{ij}(p))$ is the inverse of the matrix $(g_{ij}(p))$ whose coefficients are defined by $g_{ij}(p) = g_p(e_i, e_j)$ using a base $\{e_1, \dots, e_4\}$ of $T_p M$, $\nabla_i = \nabla_{e_i}$ and $\nabla^i = g^{ij} \nabla_j$.

Since $\Omega^0 M = C^\infty(M, \Lambda^0 M) = C^\infty(M)$, the Klein-Gordon field in our description is actually a real valued smooth function, as it is in the usual approach. The equivalence of the equations is a special case of a more general formula by Lichnerowicz (cfr. [Lic64, eq. (3.4), p. 17]) that we report here:

$$\begin{aligned} (\square_k \omega)_{i_1 \dots i_k} &= -\nabla^l \nabla_l \omega_{i_1 \dots i_k} + \sum_{n=1}^k R_{i_n l} g^{ll'} \omega_{i_1 \dots i_{n-1} l' i_{n+1} \dots i_k} \\ &\quad - \sum_{n=1}^k \sum_{n' \neq n} C_{i_n l i_{n'}}^m g^{ll'} \omega_{i_1 \dots i_{n-1} l' i_{n+1} \dots i_{n'-1} m i_{n'+1} \dots i_k}, \end{aligned} \quad (2.3.3)$$

where $\omega \in \Lambda^k M$, R_{ij} denotes the Ricci tensor and C_{ijk}^l denotes the curvature of the

Levi-Civita connection ∇ (see Subsection 1.1.2 for their definitions). This formula for $k = 0$ shows the exact coincidence of eq. (2.3.1) and eq. (2.3.2).

Remark 2.3.3. Notice that it is possible to consider also a non minimally coupled version of the Klein-Gordon equation on a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$, namely we can introduce a linear term that introduces a coupling between the field and the scalar curvature:

$$A_R \varphi = \square_0 \varphi + (m^2 + kR) \varphi = 0,$$

where k is a constant and R is the scalar curvature of the Levi-Civita connection on \mathcal{M} (see the end of Subsection 1.1.2). Indeed A_R is still a formally selfadjoint normally hyperbolic operator since we added a linear term of 0-th order in the derivatives, hence we can again apply the general construction of Section 2.2.

2.3.2 The Proca field

At a classical level we describe the Proca field of mass $m > 0$ on a 4-dimensional globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ as a 1-form $\Theta \in \Omega^1 M$ satisfying the equation

$$\delta d\Theta + m^2 \Theta = 0. \quad (2.3.4)$$

Remark 2.3.4. The standard expression in index notation for the equation of the minimally coupled Proca field on \mathcal{M} is the following:

$$-\nabla^i \nabla_i \Theta_j + \nabla^i \nabla_j \Theta_i + m^2 \Theta_j = 0, \quad (2.3.5)$$

where ∇ is the Levi-Civita connection. To check that our formulation (eq. (2.3.4)) is equivalent to the standard one (eq. (2.3.5)) we need to rewrite the standard equation in a convenient form. The first step consists in the observation that

$$\nabla^i \nabla_j \Theta_i - \nabla_j \nabla^i \Theta_i = g^{ik} R_{ij} \Theta_k.$$

This result is obtained through the direct computation of $\nabla_i \nabla_j \Theta^k$ and using the expression of the Ricci tensor R_{ij} for the Levi-Civita connection in terms of the Christoffel symbols (eq. (1.1.3)). The substitution of the last equation in eq. (2.3.5) gives

$$-\nabla^i \nabla_i \Theta_j + \nabla_j \nabla^i \Theta_i + g^{ik} R_{ij} \Theta_k + m^2 \Theta_j = 0$$

and, recalling the Lichnerowicz formula, eq. (2.3.3), for $k = 1$, we deduce that

$$(\square_1 \Theta)_j + \nabla_j \nabla^i \Theta_i + m^2 \Theta_j = 0.$$

Moreover one can check that $\delta\Theta = -\nabla^i\Theta_i$ and hence

$$\nabla_j\nabla^i\Theta_i = \partial_j\nabla^i\Theta_i = -(\mathrm{d}\delta\Theta)_j.$$

With this we conclude that eq. (2.3.4) and (2.3.5) are actually equivalent.

The case of the Proca field is more involved if compared to the case of the Klein-Gordon field. The difficulty arises at a classical level because, although being a formally selfadjoint linear differential operator of second order on $\Lambda^1 M$ (as one might easily check from eq. (2.3.5) and exploiting the fact that d and δ are formal adjoints of each other), $\delta\mathrm{d}$ is not normally hyperbolic (another glance at eq. (2.3.5) shows that the term $\nabla^i\nabla_j\Theta_i$ breaks normal hyperbolicity). This fact makes the results of Subsection 2.2.1 inapplicable to the current problem. Anyway one might observe that, since the Proca field Θ must satisfy eq. (2.3.4), then it follows that it must also be coclosed, i.e. $\delta\Theta = 0$, because $m > 0$ and

$$m^2\delta\Theta = \delta(\delta\mathrm{d}\Theta + m^2\Theta) = 0,$$

where we used the property $\delta^2 = 0$. But then $\mathrm{d}\delta\Theta = 0$ too and so the Proca field Θ satisfies also the equation

$$\square_1\Theta + m^2\Theta = 0.$$

There is no doubt that $\square_1 + m^2\mathrm{id}_{\Omega^1 M}$ is a formally selfadjoint normally hyperbolic operator and that we could apply the procedure of Subsection 2.2.1 if we consider this operator. The problem is that, proceeding in this way, we do not describe the Proca field because equation $\square_1\Theta + m^2\Theta = 0$ does not imply $\delta\Theta = 0$. However the system

$$\begin{cases} \square_1\Theta + m^2\Theta &= 0 \\ \delta\Theta &= 0 \end{cases} \quad (2.3.6)$$

is absolutely equivalent to eq. (2.3.4) as one immediately realizes. We already know how to obtain the solutions of all the Cauchy problems with compactly supported initial data formulated using the first equation of the system above. The trick that allows us to select only those solutions that satisfy also the second equation can be found in [Dap11, p. 9]. Before we present it, a lemma is required.

Lemma 2.3.5. *Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be a d -dimensional globally hyperbolic space-time, let $m \geq 0$ and let $k \in \{0, \dots, d\}$. Consider the advanced/retarded Green operator $e_k^{a/r}$ for the formally selfadjoint normally hyperbolic operator*

$$P_k = \square_k + m^2\mathrm{id}_{\Omega^k M} : \Omega^k M \rightarrow \Omega^k M.$$

We have that for each $\theta \in \Omega_0^k M$ the following identities hold:

$$\begin{aligned} e_{k+1}^{a/r}(\mathrm{d}\theta) &= \mathrm{d}\left(e_k^{a/r}\theta\right) \quad \text{for } k \in \{0, \dots, d-1\}; \\ e_{k-1}^{a/r}(\delta\theta) &= \delta\left(e_k^{a/r}\theta\right) \quad \text{for } k \in \{1, \dots, d\}. \end{aligned}$$

Proof. Fix $k \in \{0, \dots, d-1\}$ and $\theta \in \Omega_0^k M$ and consider $e_{k+1}^a \mathrm{d}\theta$. From the properties of e_k^a we know that $P_k e_k^a \theta = \theta$ so that

$$e_{k+1}^a(\mathrm{d}\theta) = e_{k+1}^a(\mathrm{d}P_k(e_k^a \theta)).$$

From Proposition 2.3.1 we deduce that $\mathrm{d} \circ P_{k+1} = P_k \circ \mathrm{d}$ and hence we find

$$e_{k+1}^a(\mathrm{d}\theta) = e_{k+1}^a(P_{k+1} \mathrm{d}(e_k^a \theta)).$$

Note that, exploiting the support properties of e_k^a , we obtain

$$\mathrm{supp}(\mathrm{d}(e_k^a \theta)) \subseteq \mathrm{supp}(e_k^a \theta) \subseteq J_+^{\mathcal{M}}(\mathrm{supp}(\theta)).$$

This inclusion implies that the support of $\mathrm{d}(e_k^a \theta)$ is \mathcal{M} -past compact because also $J_+^{\mathcal{M}}(\mathrm{supp}(\theta))$ is \mathcal{M} -past compact (cfr. Proposition 1.2.18). Then we can apply Lemma 1.3.17 to e_{k+1}^a and conclude that

$$e_{k+1}^a(\mathrm{d}\theta) = \mathrm{d}(e_k^a \theta).$$

The proof for δ in place of d is identical and we can proceed similarly also if we consider the retarded Green operators in place of the advanced ones. \square

Now we are ready to show the trick. The first step consists in the determination of the advanced and retarded Green operators for the operator $\delta \mathrm{d} + m^2 \mathrm{id}_{\Omega^k M}$. Although we cannot apply Corollary 1.3.16 because the operator is not normally hyperbolic, we can exploit the advanced and retarded Green operators for $\square_k + m^2 \mathrm{id}_{\Omega^k M}$ to find advanced and retarded Green operators for $\delta \mathrm{d} + m^2 \mathrm{id}_{\Omega^k M}$.

Lemma 2.3.6. *Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be a d -dimensional globally hyperbolic space-time and let $k \in \{1, \dots, d-1\}$. Consider the formally selfadjoint linear differential operator of second order*

$$A_k = \delta \mathrm{d} + m^2 \mathrm{id}_{\Omega^k M} : \Omega^k M \rightarrow \Omega^k M.$$

Then we have that

$$f_k^{a/r} = e_k^{a/r} \circ \left(\mathrm{id}_{\Omega_0^k M} + \frac{1}{m^2} \mathrm{d}\delta \right) : \Omega_0^k M \rightarrow \Omega^k M$$

is an advanced/retarded Green operator for A_k , where $e_k^{a/r}$ is the advanced/retarded Green operator for the formally selfadjoint normally hyperbolic operator

$$P_k = \square_k + m^2 \text{id}_{\Omega^k M} : \Omega^k M \rightarrow \Omega^k M.$$

Moreover $f_k^{r/a}$ is formally adjoint to $f_k^{a/r}$.

Proof. Fix $k \in \{1, \dots, d-1\}$. We consider only the case of the advanced Green operator (the other case being similar). First of all we notice that f_k^a is linear and that for each $\theta \in \Omega_0^1 M$ we find

$$\text{supp}(f_k^a \theta) \subseteq J_+^{\mathcal{M}} \left(\text{supp} \left(\theta + \frac{1}{m^2} d\delta \theta \right) \right) \subseteq J_+^{\mathcal{M}}(\text{supp}(\theta))$$

exploiting the support property of the advanced Green operator e_k^a . Now fix an arbitrary $\theta \in \Omega_0^k M$ and evaluate $A_k(f_k^a \theta)$ bearing in mind Lemma 2.3.5 and the properties of the Green operators:

$$\begin{aligned} A_k(f_k^a \theta) &= (\delta d + m^2 \text{id}_{\Omega^k M}) e_k^{a/r} \left(\text{id}_{\Omega_0^k M} + \frac{1}{m^2} d\delta \right) \theta \\ &= (\delta d + m^2 \text{id}_{\Omega^k M}) \left(\text{id}_{\Omega^k M} + \frac{1}{m^2} d\delta \right) e_k^{a/r} \theta \\ &= (\delta d + m^2 \text{id}_{\Omega^k M} + d\delta) e_k^{a/r} \theta \\ &= P_k(e_k^{a/r} \theta) \\ &= \theta. \end{aligned}$$

The calculation is even simpler for $f_k^a(A_k \theta)$:

$$\begin{aligned} f_k^a A_k \theta &= e_k^{a/r} \left(\text{id}_{\Omega_0^k M} + \frac{1}{m^2} d\delta \right) (\delta d + m^2 \text{id}_{\Omega^k M}) \theta \\ &= e_k^{a/r} (\square_k + m^2 \text{id}_{\Omega^k M}) \theta \\ &= \theta. \end{aligned}$$

Then we recognize that f_k^a is an advanced Green operator for A_k (cfr. Definition 1.3.15).

To conclude the proof we must show that $f_k^{r/a}$ is formally adjoint to $f_k^{a/r}$, which is to say that

$$(f_k^{r/a} \theta, \zeta)_{g,k} = (\theta, f_k^{a/r} \zeta)_{g,k}$$

for each $\theta, \zeta \in \Omega_0^k M$, where $(\cdot, \cdot)_{g,k}$ is the map defined in Proposition 1.1.46. Therefore fix θ and ζ in $\Omega_0^k M$ and evaluate $(f_k^r \theta, \zeta)_{g,k}$. Recall that $e_k^{r/a}$ is formally adjoint

to $e_k^{a/r}$ because P_k is formally selfadjoint (cfr. Proposition 1.3.21), hence we find

$$\begin{aligned} (f_k^{r/a}\theta, \zeta)_{g,k} &= \left(e_k^{r/a} \left(\text{id}_{\Omega_0^k M} + \frac{1}{m^2} d\delta \right) \theta, \zeta \right)_{g,k} \\ &= \left(\left(\text{id}_{\Omega_0^k M} + \frac{1}{m^2} d\delta \right) \theta, e^{a/r} \zeta \right)_{g,k}. \end{aligned}$$

We know also that d and δ are formal adjoints of each other (see Proposition 1.1.47) and then we can proceed in our calculation:

$$(f_k^{r/a}\theta, \zeta)_{g,k} = \left(\theta, \left(\text{id}_{\Omega^k M} + \frac{1}{m^2} d\delta \right) e^{a/r} \zeta \right)_{g,k}.$$

In the last step we exploit Lemma 2.3.5:

$$(f_k^{r/a}\theta, \zeta)_{g,k} = \left(\theta, e^{a/r} \left(\text{id}_{\Omega_0^k M} + \frac{1}{m^2} d\delta \right) \zeta \right)_{g,k} = \left(\theta, f_k^{a/r} \zeta \right)_{g,k}.$$

□

At this point we have the advanced and retarded Green operators for the operator $A_k = \delta d + m^2 \text{id}_{\Omega^k M}$ and we know that they are formal adjoints of each other. We want to use them to determine the space V of the solutions to all homogeneous Cauchy problems for the operator A_k with compactly supported initial data. Then we want to exploit the reciprocal formal adjointness of the Green operators for A_k to define a symplectic form on V .

Proposition 2.3.7. *Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be a d -dimensional globally hyperbolic spacetime and let $k \in \{1, \dots, d-1\}$. Consider the operator $A_k = \delta d + m^2 \text{id}_{\Omega^k M}$ and its advanced/retarded Green operators $f_k^{a/r}$ provided by Proposition 2.3.6. Denote with $f_k = f_k^a - f_k^r$ the corresponding causal propagator. Then the space V of the solutions to all homogeneous Cauchy problems for the operator A_k with compactly supported initial data coincides with the image through f_k of $\Omega_0^k M$, while the kernel of the causal propagator f_k coincides with the image through A_k of $\Omega_0^k M$:*

$$V = f_k(\Omega_0^k M) \quad \text{and} \quad \ker f_k = A_k(\Omega_0^k M).$$

Moreover the map

$$\begin{aligned} \sigma : V \times V &\rightarrow \mathbb{R} \\ (\Theta, \Pi) &\mapsto (f_k \Theta, \Pi)_{g,k}, \end{aligned}$$

where $(\cdot, \cdot)_{g,k}$ is the map defined in Proposition 1.1.46 and $\theta, \pi \in \Omega_0^k M$ are such that $f_k \theta = \Theta$ and $f_k \pi = \Pi$, is well defined, bilinear, non degenerate and antisymmetric, i.e. it is a symplectic form on V , hence (V, σ) is a symplectic space.

Proof. We start from the inclusion $f_k(\Omega_0^k M) \subseteq V$. Take $\Theta \in f_k(\Omega_0^k M)$ and consider $\theta \in \Omega_0^k M$ such that $f_k \theta = \Theta$. As a consequence of Lemma 2.3.6 we have that Θ is also an element of $e_k(\Omega_0^k M)$, where e_k denotes the causal propagator for $P_k = \square_k + m^2 \text{id}_{\Omega^k M}$. Therefore from Corollary 1.3.19 we deduce that Θ is the solution of a homogeneous Cauchy problem for the normally hyperbolic operator P_k with compactly supported initial data. Since we have shown that eq. (2.3.4) is equivalent to eq. (2.3.6), it is sufficient to prove that $\delta\Theta = 0$ to conclude that the expected inclusion holds. We try to evaluate $\delta\Theta$ exploiting Lemma 2.3.6 and Lemma 2.3.5:

$$\begin{aligned} \delta\Theta &= \delta \left(e_k \left(\text{id}_{\Omega_0^k M} + \frac{1}{m^2} d\delta \right) \theta \right) = e_k \left(\text{id}_{\Omega_0^k M} + \frac{1}{m^2} \delta d \right) \delta\theta \\ &= \frac{1}{m^2} e_k (m^2 + \delta d + d\delta) \delta\theta = \frac{1}{m^2} e_k P_k (\delta\theta) \\ &= 0. \end{aligned}$$

We turn our attention to the converse inclusion $V \subseteq f_k(\Omega_0^k M)$. To this end take $\Theta \in V$. Since eq. (2.3.4) is equivalent to eq. (2.3.6), Θ is also a coclosed solution of a homogeneous Cauchy problem for the normally hyperbolic operator P_k with compactly supported initial data. Applying Corollary 1.3.19, we find $\theta \in \Omega_0^k M$ such that $e_k \theta = \Theta$. Consider now $f_k \theta$ bearing in mind that $\delta\Theta = 0$ and exploiting Lemma 2.3.6 and Lemma 2.3.5:

$$f_k \theta = e_k \left(\text{id}_{\Omega_0^k M} + \frac{1}{m^2} d\delta \right) \theta = \left(\text{id}_{\Omega^k M} + \frac{1}{m^2} d\delta \right) e_k \theta = \Theta + \frac{1}{m^2} d\delta\Theta = \Theta.$$

This implies that $\Theta \in f_k(\Omega_0^k M)$.

Now we show that $\ker f_k = A_k(\Omega_0^k M)$. The inclusion $A_k(\Omega_0^k M) \subseteq \ker f_k$ is a trivial consequence of the properties of the Green operators f_k^a and f_k^r . To prove the other inclusion take θ in $\ker f_k$. This implies that $f_k^a \theta = f_k^r \theta$ and that θ is an element of $\Omega_0^k M$, hence in particular

$$\text{supp}(f_k^a \theta) \subseteq J_+^{\mathcal{M}}(\text{supp}(\theta)) \cap J_-^{\mathcal{M}}(\text{supp}(\theta)).$$

Exploiting Proposition 1.2.18, we realize that $\text{supp}(f_k^a \theta)$ is a closed subset of M included in a compact subset of M , hence it is compact too. This shows that $\theta' = f_k^a \theta \in \Omega_0^k M$ and, evaluating $A_k \theta'$ we see that $A_k \theta' = A_k(f_k^a \theta) = \theta$. Therefore we conclude that $\theta \in A_k(\Omega_0^k M)$.

The proof of the last part of this proposition is identical to the proof of Lemma 2.2.3. \square

We have associated a symplectic space (V, σ) to each triple $(\mathcal{M}, \Lambda^k M, A_k)$, where $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ is a globally hyperbolic spacetime, $\Lambda^k M$ is endowed with the inner product $\langle \cdot, \cdot \rangle_{g,k}$ induced by g (cfr. Proposition 1.1.40) and $A_k = \delta d + m^2 \text{id}_{\Omega^k M}$.

Before we proceed with the ingredients needed for the construction of a covariant

functor describing the classical theory of the Proca field, we want to introduce the category that we use as domain.

Definition 2.3.8. For $k \in \{1, \dots, d-1\}$, \mathbf{ghs}^P is the category whose objects are triples $(\mathcal{M}, \Lambda^k M, A_k)$, where $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ is a d -dimensional globally hyperbolic spacetime, $\Lambda^k M$ is endowed with the inner product induced by g and $A_k = \delta d + m^2 \text{id}_{\Omega^k M}$, whose morphisms are vector bundle homomorphisms (ψ, Ψ) from $\Lambda^k M$ to $\Lambda^k N$ over some morphism ψ from \mathcal{M} to \mathcal{N} of \mathbf{ghs} that are compatible with the inner products and the linear differential operators δd and $d\delta$ of both the domain and the codomain (for the meaning of this condition see the comments after Definition 2.2.1 and bear in mind that now normal hyperbolicity does not hold). As for the composition law, it is the usual composition of functions.

Except for the fact that the operators considered are not normally hyperbolic, \mathbf{ghs}^P can be considered as a (possibly non full) subcategory of \mathbf{ghs}^f . This claim becomes precise if we replace in all the objects of this category the operator A_k with the formally selfadjoint normally hyperbolic operator $\square_k + m^2 \text{id}_{\Omega^k M}$ because the condition of compatibility with both δd and $d\delta$ of domain and codomain entails also compatibility with $\square_k + m^2 \text{id}$ of domain and codomain. As a consequence of this fact all the conclusion that we have drawn for the morphism of \mathbf{ghs}^f hold also for the morphisms of \mathbf{ghs}^P (and maybe these morphisms have even richer properties since the compatibility condition seems to be more stringent).

Note that compatibility with δd trivially implies also compatibility with the operators $\delta d + m^2 \text{id}$ of both the domain and the codomain.

Now that we have specified the (restricted) class of morphisms that we are going to take into account we can proceed further.

Proposition 2.3.9. *Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ and $\mathcal{N} = (N, h, \mathfrak{p}, \mathfrak{u})$ be d -dimensional globally hyperbolic spacetimes, let (ψ, Ψ) be a morphism of \mathbf{ghs}^P from $(\mathcal{M}, \Lambda^k M, A_k)$ to $(\mathcal{N}, \Lambda^k N, B_k)$ and let $k \in \{1, \dots, d-1\}$. Consider the advanced/retarded Green operators $f_{k,M}^{a/r}$ and $f_{k,N}^{a/r}$ for A_k and respectively B_k provided by Lemma 2.3.6. Denote with (V, σ) and (W, ω) the symplectic spaces associated respectively to the triples $(\mathcal{M}, \Lambda^k M, A_k)$ and $(\mathcal{N}, \Lambda^k N, B_k)$. Then*

$$\text{res}_\Psi \circ f_{k,N}^{a/r} \circ \text{ext}_\Psi = f_{k,M}^{a/r},$$

where the extension map is defined in eq. (2.2.1) and the restriction map is defined in Lemma 2.2.4, and the map

$$\begin{aligned} \xi : V &\rightarrow W \\ \Theta &\mapsto f_{k,N}(\text{ext}_\Psi \theta), \end{aligned}$$

where $\theta \in \Omega_0^k M$ is such that $f_{k,M}\theta = \Theta$, is well defined, linear and compatible with the symplectic forms σ and ω , i.e. it is a symplectic map from (V, σ) to (W, ω) .

Proof. The advanced/retarded Green operators $f_{k,M}^{a/r}$ and $f_{k,N}^{a/r}$ for A_k and respectively B_k provided by Lemma 2.3.6 have the following expressions:

$$\begin{aligned} f_{k,M}^{a/r} &= e_{k,M}^{a/r} \circ \left(\text{id}_{\Omega_0^k M} + \frac{1}{m^2} d\delta \right), \\ f_{k,N}^{a/r} &= e_{k,N}^{a/r} \circ \left(\text{id}_{\Omega_0^k N} + \frac{1}{m^2} d\delta \right). \end{aligned}$$

Since we know that (ψ, Ψ) is compatible with the operators $\square_k + m^2 \text{id}_{\Omega^k M}$ and $\square_k + m^2 \text{id}_{\Omega^k N}$, we can apply Lemma 2.2.4 to deduce that

$$\text{res}_\Psi \circ e_{k,N}^{a/r} \circ \text{ext}_\Psi = e_{k,M}^{a/r},$$

while the condition of compatibility with $d\delta : \Omega^k M \rightarrow \Omega^k M$ and $d\delta : \Omega^k N \rightarrow \Omega^k N$ trivially implies that

$$\text{ext}_\Psi \left(\text{id}_{\Omega_0^k M} + \frac{1}{m^2} d\delta \right) \theta = \left(\text{id}_{\Omega_0^k N} + \frac{1}{m^2} d\delta \right) (\text{ext}_\Psi \theta)$$

for each $\theta \in \Omega_0^k M$. From these facts we deduce that

$$\begin{aligned} \left(\text{res}_\Psi \circ f_{k,N}^{a/r} \circ \text{ext}_\Psi \right) \theta &= \left(\text{res}_\Psi \circ e_{k,N}^{a/r} \circ \text{ext}_\Psi \right) \left(\text{id}_{\Omega_0^k M} + \frac{1}{m^2} d\delta \right) \theta \\ &= e_{k,M}^{a/r} \left(\text{id}_{\Omega_0^k M} + \frac{1}{m^2} d\delta \right) \theta \\ &= f_{k,M}^{a/r} \theta \end{aligned}$$

for each $\theta \in \Omega_0^k M$. This shows the first part of the thesis.

The proof of the second part is identical to the proof of Lemma 2.2.5: we can proceed in the same way simply considering the Green operators for A_k and B_k provided by Lemma 2.3.6 thanks to the result of the first part of our proof. \square

Exploiting Proposition 2.3.7 and Proposition 2.3.9, we can introduce the covariant functor describing the classical theory of the Proca field as shown by the next theorem. In the following we choose the dimension of the spacetimes $d = 4$ and we drop the subscript k since we fix $k = 1$. However note that the theorem holds also for each $d \in \mathbb{N}$ and each $k \in (1, \dots, d-1)$. The choices $d = 4$ and $k = 1$ are made for compatibility with the physical problem.

Theorem 2.3.10. *Consider the map*

$$\begin{aligned} \mathcal{B} : \quad \text{Obj}_{\text{ghs}^P} &\rightarrow \text{Obj}_{\text{ssp}} \\ (\mathcal{M}, \Lambda^1 M, A) &\mapsto (V, \sigma) \end{aligned}$$

defined following Proposition 2.3.7 and for each pair $(\mathcal{M}, \Lambda^1 M, A)$, $(\mathcal{N}, \Lambda^1 N, B)$ in $\text{Obj}_{\text{ghs}^P}$ consider the map

$$\begin{aligned} \mathcal{B} : \text{Mor}_{\text{ghs}^P}((\mathcal{M}, \Lambda^1 M, A), (\mathcal{N}, \Lambda^1 N, B)) &\rightarrow \text{Mor}_{\text{ssp}}((V, \sigma), (W, \omega)) \\ (\psi, \Psi) &\mapsto \xi \end{aligned}$$

defined in accordance with Proposition 2.3.9, where (V, σ) and (W, ω) respectively denote the symplectic spaces $\mathcal{B}(\mathcal{M}, \Lambda^1 M, A)$ and $\mathcal{B}(\mathcal{N}, \Lambda^1 N, B)$. These maps give rise to a covariant functor \mathcal{B} from ghs^P to ssp that fulfils the following properties:

- *causality:* for each $(\mathcal{M}_1, \Lambda^1 M_1, A_1)$, $(\mathcal{M}_2, \Lambda^1 M_2, A_2)$, $(\mathcal{M}, \Lambda^1 M, A)$ in $\text{Obj}_{\text{ghs}^P}$, each morphism (ψ_1, Ψ_1) from $(\mathcal{M}_1, \Lambda^1 M_1, A_1)$ to $(\mathcal{M}, \Lambda^1 M, A)$ and each morphism (ψ_2, Ψ_2) from $(\mathcal{M}_2, \Lambda^1 M_2, A_2)$ to $(\mathcal{M}, \Lambda^1 M, A)$ such that $\psi_1(M_1)$ and $\psi_2(M_2)$ are \mathcal{M} -causally separated, we have that

$$\sigma(\xi_1 \Theta_1, \xi_2 \Theta_2) = 0$$

for each $\Theta_1 \in V_1$ and each $\Theta_2 \in V_2$, where (V_1, σ_1) , (V_2, σ_2) and (V, σ) denote the symplectic spaces corresponding respectively to $(\mathcal{M}_1, \Lambda^1 M_1, A_1)$, $(\mathcal{M}_2, \Lambda^1 M_2, A_2)$ and $(\mathcal{M}, \Lambda^1 M, A)$, while ξ_1 and ξ_2 denote the symplectic maps corresponding respectively to (ψ_1, Ψ_1) and (ψ_2, Ψ_2) ;

- *time slice axiom:* for each $(\mathcal{M}, \Lambda^1 M, A)$, $(\mathcal{N}, \Lambda^1 N, B)$ in $\text{Obj}_{\text{ghs}^P}$ and each morphism (ψ, Ψ) from $(\mathcal{M}, \Lambda^1 M, A)$ to $(\mathcal{N}, \Lambda^1 N, B)$ such that $\psi(M)$ includes a smooth spacelike Cauchy surface Σ for \mathcal{N} , we have that

$$\xi(V) = W,$$

where (V, σ) and (W, ω) denote the symplectic spaces corresponding respectively to $(\mathcal{M}, \Lambda^1 M, A)$ and $(\mathcal{N}, \Lambda^1 N, B)$, while ξ denotes the symplectic map corresponding to (ψ, Ψ) .

Proof. The check of the covariant axioms, as well as the proof of the causality property, are identical to those in the proof of Theorem 2.2.6. For the proof of the time slice axiom again we can largely imitate the proof of the above mentioned theorem. We must only remember to write the Cauchy problem used to pick out the compact subset K for the normally hyperbolic operator $\square + m^2 \text{id}_{\Omega^1 M}$ in place of the operator B (which is not normally hyperbolic), otherwise we cannot apply Theorem 1.3.7 to deduce uniqueness of the solution. The use of $\square + m^2 \text{id}_{\Omega^1 M}$ in place of B does not give rise to problems because sections $\Theta \in \Omega^1 N$ satisfying $B\Theta = 0$ also satisfy $\square\Theta + m^2\Theta = 0$ (bear in mind the equivalence between eq. (2.3.4) and eq. (2.3.6)). At a certain point of the proof we should find an identity of the type

$B\Theta^+ = -B\Theta^-$. This entails $\delta\Theta^+ = -\delta\Theta^-$ and hence also

$$(\square + m^2 \text{id}_{\Omega^1 M}) \Theta^+ = -(\square + m^2 \text{id}_{\Omega^1 M}) \Theta^-.$$

As we deduce from $B\Theta^+ = -B\Theta^-$ that $B\Theta^+$ is a section of $\Omega_0^1 N$ with support included in $\psi(M)$, in a similar manner we deduce from the equation above that $(\square + m^2 \text{id}_{\Omega^1 M}) \Theta^+$ is a section of $\Omega_0^1 N$ with support included in $\psi(M)$. Towards the end we resorted to Lemma 1.3.17. This is not directly applicable in the present situation because B is not normally hyperbolic, however it holds that

$$f_N^a(B\Theta^+) = e_N^a(\square + m^2 \text{id}_{\Omega^1 M}) \Theta^+$$

where e_N^a denotes the advanced/retarded Green operator for the normally hyperbolic operator $\square + m^2 \text{id}_{\Omega^1 M}$. Since we have just shown that $(\square + m^2 \text{id}_{\Omega^1 M}) \Theta^+$ has compact support, we are again in position to apply Lemma 1.3.17. A similar procedure applies to $f_N^r(B\Theta^-)$ and this leads us to the end of the proof. \square

With the last theorem we have completed the classical theory of the Proca field. Now we must proceed with the quantization of the classical theory that can be done composing our functor $\mathcal{B} : \mathbf{ghs}^P \rightarrow \mathbf{ssp}$ with the functor $\mathcal{C} : \mathbf{ssp} \rightarrow \mathbf{alg}$ built in Lemma 2.2.7. As we proved in Theorem 2.2.8, the result is a locally covariant quantum field theory $\mathcal{A} = \mathcal{C} \circ \mathcal{B}$ that satisfies both the causality condition and the time slice axiom. In turn this implies that we can apply Theorem 2.1.9 and Remark 2.2.9. Therefore on each globally hyperbolic spacetime \mathcal{A} provides the quantum field theory of the Proca field according to the algebraic approach suggested by Haag and Kastler.

Before we pass to the last example, we want to make some remarks about the morphisms that are usually taken into account when dealing with the realization of a LCQFT for a field of physical interest, such as the Klein-Gordon field or the Proca field.

Remark 2.3.11. In the discussion of the classical theory of a concrete field, for example the Klein-Gordon field or the Proca field, it is usual to consider only push-forwards and pull-backs as vector bundle homomorphisms. For the case of the Klein-Gordon field we noted that these two approaches are equivalent. We show now that push-forwards and pull-backs are morphisms of \mathbf{ghs}^P so that our approach surely includes the usual one.

Note that if ψ is an orientation and time orientation preserving isometric diffeomorphism from $\mathcal{M} = (M, g, \mathbf{o}, \mathbf{t})$ to $\mathcal{N} = (N, h, \mathbf{p}, \mathbf{u})$, we realize immediately that (ψ, ψ_*) is a bijective morphism of \mathbf{ghs}^P between the objects $(\mathcal{M}, \Lambda^k M, A_k)$ and $(\mathcal{N}, \Lambda^k N, B_k)$ whose inverse (ψ^{-1}, ψ^*) is a morphism of \mathbf{ghs}^f too: $\psi_* : \Lambda^k M \rightarrow \Lambda^k N$ is defined as the pull-back through ψ^{-1} (see. Remark 1.1.9) and (ψ, ψ_*) is indeed a vector bundle isomorphism from $\Lambda^k M$ to $\Lambda^k N$ (cfr. Remark 1.1.17) which is com-

patible with the inner products induced by the metrics and with the operators A_k and B_k because ψ is isometric and the following identities hold (see Proposition 1.1.34 and the comments after Definition 1.1.41):

$$\begin{aligned}\psi_* \circ d &= d \circ \psi_*, \\ \psi_* \circ \delta &= \delta \circ \psi_*.\end{aligned}$$

When ψ is only an orientation and time orientation preserving isometric embedding from $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ to $\mathcal{N} = (N, h, \mathfrak{p}, \mathfrak{u})$ whose image $\psi(M)$ is an open subset of N (i.e. a morphism of our category \mathbf{ghs}), we can apply the conclusions above to the diffeomorphism $\psi' : M \rightarrow \psi(M)$, $p \mapsto \psi(p)$ and obtain a morphism of \mathbf{ghs}^P :

$$(\psi', \psi'_*) : (\mathcal{M}, \Lambda^k M, A_k) \rightarrow (\psi(\mathcal{M}) = \mathcal{N}|_{\psi(M)}, \Lambda^k \psi(M) = \Lambda^k N|_{\psi(M)}, B_k).$$

Then we find a new morphism of \mathbf{ghs}^P from $(\mathcal{M}, \Lambda^1 M, A)$ to $(\mathcal{N}, \Lambda^1 N, B)$ defining the vector bundle homomorphism

$$\begin{aligned}(\psi, \psi_*) : \Lambda^1 M &\rightarrow \Lambda^1 N \\ (p, \omega) &\mapsto (\psi'(p), \psi'_* \omega).\end{aligned}$$

(ψ, ψ_*) inherits all the properties of (ψ', ψ'_*) with the only exception that it is not surjective and hence it is actually a morphism of \mathbf{ghs}^P . As a matter of fact we have simply defined (ψ, ψ_*) as the composition of $(\iota_{\psi(M)}^N, \iota_{\Lambda^1 \psi(M)}^{\Lambda^1 N})$ with (ψ', ψ'_*) , which are indeed morphisms of \mathbf{ghs}^P .

On the contrary one may find morphisms (ψ, Ψ) of \mathbf{ghs}^P which are not of the form (ψ, ψ_*) : For example consider the Minkowski spacetime as globally hyperbolic spacetime \mathcal{M} ; the vector bundle isomorphism $(\text{id}_{\mathbb{R}^4}, \Psi) : \Lambda^k M \rightarrow \Lambda^k M$, where Ψ acts on each fiber as a fixed Lorentz transformation L for tensors of type $(0, k)$, is a bijective morphism of \mathbf{ghs}^P from $(\mathcal{M}, \Lambda^k M, A_k)$ whose inverse is a morphism too, but it is not of the form $(\text{id}_{\mathbb{R}^4}, \text{id}_{\Lambda^k M})$ because $\text{id}_{\mathbb{R}^4} = \text{id}_{\Lambda^k M}$, where $\Lambda^k M = \mathbb{R}^4 \times \mathbb{R}^n$, $n = \binom{4}{k}$, in the present situation. This means that we are dealing with a potential enlargement of the family of morphisms usually considered (that is comprised by pull-backs and push-forwards through morphisms of \mathbf{ghs}).

We take the chance to anticipate that for the upcoming example, the electromagnetic field, we will be forced to reduce to usual approach, that is our morphisms will be only pull-backs and push-forwards through morphisms of \mathbf{ghs} .

2.3.3 The electromagnetic field

Consider a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$. The electromagnetic field is usually described by a section \mathbf{F} (known as field strength) in the vector bundle $\Lambda^2 M$, i.e. a 2-form over M , satisfying Maxwell equations

$$\begin{cases} d\mathbf{F} &= 0, \\ \delta\mathbf{F} &= 0. \end{cases}$$

If the second de Rham cohomology group is trivial, that is $H^2(M) = \{0\}$, then all closed 2-forms over M are also exact (cfr. Definition 1.1.35). This means that we can find a 1-form \mathbf{A} over M (called vector potential) such that $d\mathbf{A} = \mathbf{F}$ and the Maxwell equations reduce to

$$\delta d\mathbf{A} = 0, \tag{2.3.7}$$

that is a version of the Proca equation with $m = 0$ (cfr. eq. (2.3.4)). But when M is such that $H^2(M)$ is not trivial it happens that there are closed 2-forms \mathbf{F} such that the equation $d\mathbf{A} = \mathbf{F}$ cannot be verified by any 1-form \mathbf{A} , hence we cannot deduce eq. (2.3.7) from Maxwell equations. This means that there exist field strengths which are indeed solutions of the Maxwell equations, but are not generated by a vector potential satisfying eq. (2.3.7).

The problem in dealing directly with the Maxwell equations is the absence of a normally hyperbolic operator that allows us to apply the theory about wave equations we presented in Section 1.3. Then we are induced to the choice of an approach based on the vector potential \mathbf{A} and eq. (2.3.7) in place of the field strength and the Maxwell equations, although the essential physical observable in our description is still the field strength \mathbf{F} (not the vector potential \mathbf{A}), as it was in the approach based on the Maxwell equations. Indeed we recover the Maxwell equations simply defining $\mathbf{F} = d\mathbf{A}$, but we automatically exclude from our description all those field strengths that are not closed. In conclusion we renounce to the description of all the field strengths admitted by the Maxwell equations to obtain an equation which seems to be more convenient. However, exactly as in the case of the Proca field, δd is formally selfadjoint linear differential operator of second order, but it fails to be normally hyperbolic and hence we cannot automatically obtain advanced and retarded Green operators on each globally hyperbolic spacetime. Moreover now eq. (2.3.7) does not imply that $\delta\mathbf{A} = 0$ because of the absence of the mass term and hence the system

$$\begin{cases} \square_1 \mathbf{A} &= 0, \\ \delta\mathbf{A} &= 0 \end{cases} \tag{2.3.8}$$

is not equivalent to eq. (2.3.7), although solutions \mathbf{A} of the system are solutions of eq. (2.3.7) too. Then we cannot attempt a procedure similar to that followed for the Proca field to show that the Green operators for A are related to those for \square_1 .

Luckily there is *gauge equivalence* that comes to our aid. We said that the physical observable is the field strength $F = dA$. It may happen that different vector potentials A and A' satisfying eq. (2.3.7) generate the same field strength F , in which case they are said to be gauge equivalent. Then from a physical point of view A and A' are indistinguishable since they generate the same observable. Hence we do not want to have in our classical theory of the electromagnetic field both A and A' as distinguished dynamical configurations of the vector potential. The next lemma puts together these facts showing that eq. (2.3.7) and eq. (2.3.8) become equivalent when we identify gauge equivalent configurations.

Lemma 2.3.12. *Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be a globally hyperbolic spacetime and consider $A \in \Omega^1 M$. Then the following conditions are equivalent:*

- *A satisfies the equation $\delta dA = 0$;*
- *there exists $A' \in \Omega^1 M$, which is gauge equivalent to A , i.e. $d(A' - A) = 0$, that satisfies the equation $\square_1 A' = 0$ and the Lorentz gauge condition $\delta A' = 0$.*

Moreover consider the space S_1 of gauge inequivalent classes of 1-forms satisfying $\delta dA = 0$,

$$S_1 = \frac{\{A \in \Omega^1 M : \delta dA = 0\}}{\{A \in \Omega^1 M : dA = 0\}},$$

and the space S_2 of gauge inequivalent classes of 1-forms satisfying $\square_1 A' = 0$ and the Lorentz gauge condition,

$$S_2 = \frac{\{A' \in \Omega^1 M : \square_1 A' = 0, \delta A' = 0\}}{\{A' \in \Omega^1 M : dA' = 0, \delta A' = 0\}}.$$

Then the map $I : S_1 \rightarrow S_2$ defined by $I[A]_1 = [A']_2$, where A is a representative of the class $[A]_1$, A' , which is gauge equivalent to A , satisfies $\square_1 A' = 0$ and the Lorentz gauge condition and $[A']_2$ denotes the class that has A' as representative, is a vector space isomorphism.

Proof. Fix $A \in \Omega^1 M$. If we suppose that there exists $A' \in \Omega^1 M$ such that

$$\begin{cases} d(A' - A) = 0, \\ \square_1 A' = 0, \\ \delta A' = 0, \end{cases}$$

then we immediately deduce that

$$\delta dA = \delta dA' = \delta dA' + d\delta A' = \square_1 A' = 0.$$

Conversely suppose that $\delta dA = 0$. Consider the equation $\square_0 f = -\delta A$. In [Gin09, Cor. 5, p. 78] we find a procedure that extends the result of Theorem 1.3.7 stating

the existence and uniqueness of the solution of a Cauchy problem for a normally hyperbolic operator even when the initial data are not compactly supported. We deduce that there exists $f \in \Omega^0 M$ satisfying $\square_0 f = -\delta A$. We set $A' = A + df$ and we check that A' fulfils the requirements of the second condition in the statement of the proposition. Indeed $d(A' - A) = d(df) = 0$. Moreover, applying Proposition 2.3.1, we find

$$\square_1 A' = \square_1 A + \square_1 df = d\delta A + d\square_0 f = 0.$$

It remains to check only the Lorentz gauge condition:

$$\delta A' = \delta A + \delta df = \delta A + \square_0 f = 0.$$

Now we turn our attention to the definition of I . Take $[A]_1 \in S_1$ and consider two representatives A and B of $[A]_1$. Then $\delta dA = 0 = \delta dB$ and, applying the first part of this lemma, we find A' and B' in $\Omega^1 M$ such that

$$\left\{ \begin{array}{lcl} \square_1 A' & = & 0 = \square_1 B', \\ \delta A' & = & 0 = \delta B', \\ d(A' - A) & = & 0 = d(B' - B). \end{array} \right.$$

In particular we deduce that

$$d(A' - B') = d(A - B) = 0$$

because A and B are gauge equivalent being representatives of the same equivalence class of S_1 . Moreover trivially $\delta(A' - B') = 0$. This proves that I is well defined. Linearity can be directly checked from the definition of I . Consider now $[A]_1$ such that $I[A]_1 = [0]_2$, where $[0]_2$ denotes the class of S_2 that has the null section as representative (this is actually the zero element of the vector space S_2). This means that each representative A of the class $[A]_1$ is gauge equivalent to each representative A' of the class $[0]_2$. In particular we choose the null section 0 as representative of $[0]_2$ and we deduce that each representative A of the class $[A]_1$ is such that $dA = 0$, i.e. $[A]_1$ is the zero element of the vector space S_1 (we may write $[A]_1 = [0]_1$). Then we conclude that I is injective. To conclude the proof take $[A']_2 \in S_2$. We look for $[A]_1 \in S_1$ such that $I[A]_1 = [A']_2$. Take a representative $A \in [A']_2$. This in particular verifies $\delta dA = 0$ and therefore we can consider the class $[A]_1$ that has A as representative. Applying the definition of I , we see that $I[A]_1 = [A]_2 = [A']_2$. This shows that I is also surjective and hence it is a vector space isomorphism as expected. \square

The last theorem gives us the opportunity to identify S_1 with S_2 . This means that we can equivalently consider gauge inequivalent classes of 1-forms satisfying $\delta dA = 0$ or gauge inequivalent classes of vector potentials satisfying both $\square_1 A = 0$

and $\delta\mathbf{A} = 0$ as classical observables of the electromagnetic field.

For the construction of a covariant functor describing the classical theory of the electromagnetic field, we need to determine a symplectic space comprised by all the gauge inequivalent classes of solutions for homogeneous Cauchy problems with compactly supported initial data associated to the operator δd . This must be done for each globally hyperbolic spacetime. Unfortunately the lack of normal hyperbolicity and the presence gauge invariance significantly alter the situation of Subsection 2.2.1 so that we are forced to start the construction of the classical theory from the beginning.

In first place we try to determine the vector space on which we will define a symplectic form. The solution of this problem is suggested by [Dim92, Prop. 4, p. 228]. Note that from now on we say that \mathbf{A} is a *Lorentz 1-form* if it is a 1-form satisfying the Lorentz gauge condition, i.e. $\delta\mathbf{A} = 0$.

Lemma 2.3.13. *Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be a globally hyperbolic spacetime and define the space of compactly supported coclosed 1-forms over M :*

$$\Omega_{0,\delta}^1 M = \{ \theta \in \Omega_0^1 M : \delta\theta = 0 \} .$$

The Lorentz solution of a homogeneous Cauchy problem for the normally hyperbolic operator \square_1 with compactly supported initial data is gauge equivalent to $e\theta$, where e is the causal propagator for the normally hyperbolic operator \square_1 and θ is some element of $\Omega_{0,\delta}^1 M$. Conversely, for each $\theta \in \Omega_{0,\delta}^1 M$, $e\theta$ is a Lorentz solution of a homogeneous Cauchy problem for the normally hyperbolic operator \square_1 with compactly supported initial data.

It follows immediately that the space of gauge inequivalent classes of Lorentz solutions of homogeneous Cauchy problems for the normally hyperbolic operator \square_1 with compactly supported initial data coincides with the following subset of S_2 (for the definition of S_2 refer to Lemma 2.3.12):

$$V = \{ [e\theta]_2 : \theta \in \Omega_{0,\delta}^1 M \} \subseteq S_2,$$

where $[e\theta]_2$ denotes the class of S_2 that has $e\theta$ among its representatives.

Proof. Consider a Lorentz solution $\mathbf{A} \in \Omega^1 M$ of a homogeneous Cauchy problem for the normally hyperbolic operator \square_1 with compactly supported initial data on a given Cauchy surface Σ for M . Then we find a compact subset K of M including the support of the initial data for the Cauchy problem and we take a relatively compact open subset O of M including K . We deduce that

$$\{ J_+^{\mathcal{M}}(O), J_-^{\mathcal{M}}(O), M \setminus J^{\mathcal{M}}(K) \}$$

is an open covering of M because $J_{\pm}^{\mathcal{M}}(O)$ are open subsets of M (see [FV11, Lem.

A.8, p. 48]) and $J_{\pm}^{\mathcal{M}}(K)$ are closed subsets of M (see [BGP07, Lem. A.5.1, p. 173]) and we can introduce a partition of unity subordinate to such covering:

$$\{\chi^+, \chi^-, \chi^0\}.$$

Defining $\mathbf{A}^{\pm} = \chi^{\pm} \mathbf{A}$ and $\mathbf{A}^0 = \chi^0 \mathbf{A}$, we see that $\mathbf{A} = \mathbf{A}^+ + \mathbf{A}^- + \mathbf{A}^0$. But K includes the support of the initial data for the solution \mathbf{A} so that $\text{supp}(\mathbf{A}) \subseteq J^{\mathcal{M}}(K)$ (this is a consequence of Theorem 1.3.7, which can be applied because \square_1 is normally hyperbolic) and hence

$$\text{supp}(\mathbf{A}^0) = \text{supp}(\chi^0) \cap \text{supp}(\mathbf{A}) \subseteq (M \setminus J^{\mathcal{M}}(K)) \cap J^{\mathcal{M}}(K) = \emptyset.$$

This means that $\mathbf{A}^0 = 0$ and so $\mathbf{A} = \mathbf{A}^+ + \mathbf{A}^-$. From $\square_1 \mathbf{A} = 0$ and $\delta \mathbf{A} = 0$ we deduce that $\square_1 \mathbf{A}^+ = -\square_1 \mathbf{A}^-$ and $\delta \mathbf{A}^+ = -\delta \mathbf{A}^-$. The first of these identities implies that $\square_1 \mathbf{A}^+$ has compact support because we can apply Proposition 1.2.18 to

$$\text{supp}(\delta \mathbf{A}^+) \subseteq \text{supp}(\chi^+) \cap \text{supp}(\chi^-) \subseteq J_+^{\mathcal{M}}(\overline{O}) \cap J_-^{\mathcal{M}}(\overline{O})$$

noting that \overline{O} is a compact subset of M since by construction O is a relatively compact subset of M . A similar procedure shows also that $\delta \mathbf{A}^+$ has compact support. Then, considering $\theta = \delta \mathbf{A}^+$, we have a compactly supported 1-form that trivially satisfies $\delta \theta = 0$. We must only check that $e\theta$ is gauge equivalent to \mathbf{A} . Applying Lemma 2.3.5, we see that $d(e^a \theta) = e^a(d\theta)$. Evaluating $d\theta$ and keeping in mind that $d^2 = 0$, we obtain

$$d\theta = d\delta \mathbf{A}^+ = \square_1 d\mathbf{A}^+.$$

Proposition 1.2.18 implies that \mathbf{A}^+ has past compact support so that we can exploit Lemma 1.3.17 to obtain

$$d(e^a \theta) = e^a \square_1 d\mathbf{A}^+ = d\mathbf{A}^+.$$

A similar procedure shows that $d(e^r \theta) = -d\mathbf{A}^-$ and therefore we conclude $d(e\theta) = d\mathbf{A}$, which means exactly that $e\theta$ is gauge equivalent to \mathbf{A} .

Now take $\theta \in \Omega_{0,\delta}^1 M$ and consider $e\theta$. Trivially $\square_1(e\theta) = 0$ and by Lemma 2.3.5 we see that $\delta(e\theta) = e(\delta\theta) = 0$ ($e\theta$ is a Lorentz 1-form). To see that $e\theta$ is also a solution of a homogeneous Cauchy problem for the normally hyperbolic operator \square_1 with compactly supported initial data, we take a spacelike smooth Cauchy surface Σ for \mathcal{M} (the existence is assured by Theorem 1.2.15) and we define on it a \mathfrak{t} -future directed g -timelike unit vector field \mathbf{n} over Σ normal to Σ . Then we take α_0 as the restriction of $e\theta$ to Σ and α_1 as the restriction of $\nabla_n(e\theta)$ to Σ , where ∇ denotes the Levi-Civita connection. α_0 and α_1 are indeed sections in the restriction of $\Lambda^1 M$ to Σ and their supports are compact because we know that θ has compact support and

we can apply Proposition 1.2.18 to

$$\text{supp}(e\theta) \cap \Sigma \subseteq J^{\mathcal{M}}(\text{supp}(\theta)) \cap \Sigma.$$

Then we can consider the following Cauchy problem:

$$\begin{cases} \square_1 \mathbf{A} &= 0, \\ \mathbf{A}|_{\Sigma} &= \alpha_0, \\ \nabla_{\mathbf{n}} \mathbf{A}|_{\Sigma} &= \alpha_1. \end{cases}$$

By construction $e\theta$ is a solution (actually the unique solution due to Theorem 1.3.7, which holds because \square_1 is normally hyperbolic). Since we have shown at the beginning of the proof that $\delta(e\theta) = 0$, we conclude that $e\theta$ is a Lorentz solution of a homogeneous Cauchy problem for the normally hyperbolic operator \square_1 with compactly supported initial data. This completes the proof. \square

We have a vector space V . Now we need a symplectic form on it. A new difficulty associated to the first de Rham cohomology group of M arises in this situation as we will see in the proof of the next Lemma. To go around this obstacle we assume that $H^1(M) = \{0\}$ following the approach of [Dap11].

Lemma 2.3.14. *Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be a globally hyperbolic spacetime such that $H^1(M) = \{0\}$ and consider the vector space V defined in Lemma 2.3.13. The map*

$$\begin{aligned} \sigma : \quad V \times V &\rightarrow \mathbb{R} \\ ([\mathbf{A}]_2, [\mathbf{B}]_2) &\mapsto (e\theta, \zeta)_{g,1}, \end{aligned}$$

where e is the causal propagator for the formally selfadjoint normally hyperbolic operator \square_1 , θ and ζ are elements of $\Omega_{0,\delta}^1 M$ such that $e\theta$ and $e\zeta$ are representatives of $[\mathbf{A}]_2$ and respectively $[\mathbf{B}]_2$ and $(\cdot, \cdot)_{g,1}$ is defined in Proposition 1.1.46, is well defined, bilinear, antisymmetric and non degenerate, i.e. it is a symplectic map on V . Hence (V, σ) is a symplectic space.

Proof. To show that σ is well defined, take $[\mathbf{A}]_2$ and $[\mathbf{B}]_2$ in V . Because of the definition of V we find θ and ζ in $\Omega_{0,\delta}^1 M$ such that $e\theta \in [\mathbf{A}]_2$ and $e\zeta \in [\mathbf{B}]_2$. Since ζ is compactly supported, we can evaluate $(e\theta, \zeta)_{g,1}$ and indeed we get a real number. If we consider also θ' and ζ' in $\Omega_{0,\delta}^1 M$ such that $e\theta' \in [\mathbf{A}]_2$ and $e\zeta' \in [\mathbf{B}]_2$, we have $(e\theta', \zeta')_{g,1}$ and we must check that it coincides with $(e\theta, \zeta)_{g,1}$ in order to have σ well defined. Since $e\theta$ and $e\theta'$ are both representatives of $[\mathbf{A}]_2$, they are gauge equivalent, i.e. $d(e\theta - e\theta') = 0$. Now the hypothesis $H^1(M) = \{0\}$ comes into play because it implies that we can find $\alpha \in \Omega^0 M$ such that $e\theta - e\theta' = d\alpha$. Similarly we find $\beta \in \Omega^0 M$ such that $e\zeta - e\zeta' = d\beta$. bearing in mind that e is formally antiselfadjoint (because \square_1 is formally selfadjoint, cfr. Proposition 1.3.21) and that d and δ are

formal adjoints of each other, we can tackle the evaluation of $(e\theta', \zeta')_{g,1}$:

$$\begin{aligned}
(e\theta', \zeta')_{g,1} &= (e\theta, \zeta')_{g,1} - (d\alpha, \zeta')_{g,1} \\
&= -(\theta, e\zeta')_{g,1} - (\alpha, \delta\zeta')_{0,g} \\
&= -(\theta, e\zeta)_{g,1} + (\theta, d\beta)_{g,1} \\
&= (e\theta, \zeta)_{g,1} + (\delta\theta, \beta)_{g,0} \\
&= (e\theta, \zeta)_{g,1},
\end{aligned}$$

where we exploited $\delta\zeta' = 0$ and $\delta\beta = 0$. This shows that σ is well defined. Notice that without the hypothesis $H^1(M) = \{0\}$ this proof does not work.

Bilinearity of σ easily follows from bilinearity of $(\cdot, \cdot)_{g,1}$ and linearity of e . As for antisymmetry, consider $[A]_2$ and $[B]_2$ in V . By definition of V we find θ and ζ in $\Omega_{0,\delta}^1 M$ such that $e\theta \in [A]_2$ and $e\zeta \in [B]_2$. Exploiting the definition of σ , the antiselfadjointness of e and the symmetry of $(\cdot, \cdot)_{g,1}$, we obtain

$$\sigma([A]_2, [B]_2) = (e\theta, \zeta)_{g,1} = -(\theta, e\zeta)_{g,1} = -(e\zeta, \theta)_{g,1} = -\sigma([B]_2, [A]_2).$$

It remains only to check that σ is non degenerate. To this end consider $[A]_2 \in V$ such that $\sigma([A]_2, [B]_2) = 0$ for each $[B]_2 \in V$. Taking $\theta \in \Omega_{0,\delta}^1 M$ such that $e\theta \in [A]_2$, we deduce that $(e\theta, \zeta)_{g,1} = 0$ for each $\zeta \in \Omega_{0,\delta}^1 M$. In particular we have $(e\theta, \delta\alpha)_{g,1} = 0$ for each $\alpha \in \Omega_0^2 M$ and hence $(d(e\theta), \alpha)_{g,2} = 0$ for each $\alpha \in \Omega_0^2 M$. Since $(\cdot, \cdot)_{g,2}$ is non degenerate, we conclude that $d(e\theta) = 0$. This fact means that $e\theta$ is a representative of the zero class of V , i.e. $[A]_2 = [0]_2$. \square

At this point we are able to associate a symplectic space (V, σ) comprised by all the gauge inequivalent classes of dynamical configuration for the electromagnetic field on each globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ such that the first de Rham cohomology group of M is trivial.

This fact induces us to introduce of a special category \mathbf{ghs}^{EM} for the electromagnetic field.

Definition 2.3.15. We define the category \mathbf{ghs}^{EM} in the following way:

- objects are triples $(\mathcal{M}, \Lambda^1 M, \delta d)$, where $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ is a globally hyperbolic spacetime with $H^1(M) = \{0\}$, $\Lambda^1 M$ is the vector bundle over M that we consider, $\langle \cdot, \cdot \rangle_{g,1}$ is the inner product on $\Lambda^1 M$ induced by the metric g (refer to Proposition 1.1.40 for a characterization of this inner product) and δd is the linear differential operator on $\Lambda^1 M$ over \mathcal{M} governing the dynamics of the electromagnetic field;
- morphisms from $(\mathcal{M}, \Lambda^1 M, \delta d)$ to $(\mathcal{N}, \Lambda^1 N, \delta d)$ are vector bundle homomorphisms of the form (ψ, ψ_*) from $\Lambda^1 M$ to $\Lambda^1 N$ such that ψ is a morphism of

\mathfrak{ghs} (note that $H^1(M) = \{0\}$ entails $H^1(\psi(M)) = \{0\}$);¹

- the composition law is simply the composition of functions.

This is a specialization of the category \mathfrak{ghs}^f , actually a subcategory (but not a full subcategory because we consider only push-forwards). This statement is not at all correct because here δd is not normally hyperbolic, but it becomes rigorous if we replace δd with the normally hyperbolic operator \square_1 . Then all the observations referred to \mathfrak{ghs}^f hold also for \mathfrak{ghs}^{EM} .

Applying Lemma 2.3.13 and Lemma 2.3.14, we can define the map

$$\begin{aligned} \mathcal{B} : \quad \text{Obj}_{\mathfrak{ghs}^{EM}} &\rightarrow \text{Obj}_{\mathfrak{ssp}} \\ (\mathcal{M}, \Lambda^1 M, \delta d) &\mapsto (V, \sigma). \end{aligned}$$

This is the first part of our candidate functor describing the classical theory of the electromagnetic field. The second part comes from the next lemma.

Before presenting the statement, we introduce some notation. From now on the vector space that was denoted by S_2 in Lemma 2.3.12 will be denoted by S_M to keep trace of the manifold we are working on. Similarly the equivalence class previously indicated with $[\cdot]_2$ will be denoted by $[\cdot]_M$.

Lemma 2.3.16. *Let (ψ, ψ_*) be a morphism of \mathfrak{ghs}^{EM} from the object $(\mathcal{M}, \Lambda^1 M, \delta d)$ to the object $(\mathcal{N}, \Lambda^1 N, \delta d)$. Denote with (V, σ) and (W, ω) the symplectic spaces associated to $(\mathcal{M}, \Lambda^1 M, \delta d)$ and respectively $(\mathcal{N}, \Lambda^1 N, \delta d)$ by the map \mathcal{B} defined few lines above. Then the map*

$$\begin{aligned} \xi : \quad V &\rightarrow W \\ [\mathbf{A}]_M &\mapsto [e_N(\text{ext}_{\psi_*}\theta)]_N, \end{aligned}$$

where e_M and e_N are the causal propagators for the formally selfadjoint normally hyperbolic operator \square_1 on $\Lambda^1 M$ over \mathcal{M} and respectively on $\Lambda^1 N$ over \mathcal{N} and $\theta \in \Omega_{0,\delta}^1 M$ is such that $e_M \theta \in [\mathbf{A}]_M$, is well defined, linear and compatible with the symplectic forms σ and ω , that is to say that ξ is a symplectic map from (V, σ) to (W, ω) .

Proof. The first step of this proof is devoted to show that ξ is well defined. To this end take $[\mathbf{A}]_M \in V$. By definition we find $\theta \in \Omega_{0,\delta}^1 M$ such that $e_M \theta \in [\mathbf{A}]_M$. It

¹For the electromagnetic field we are forced to restrict our class of morphisms to that usually considered, i.e. only vector bundle homomorphisms that are push-forwards of morphisms of \mathfrak{ghs} . For the precise definition of these morphisms and some comments refer to Remark 2.3.11. This choice is done because push-forwards (and similarly pull-backs) have the property of being defined on k -forms for arbitrary k and moreover they intertwine with both d and δ (see Proposition 1.1.34 and comments after Definition 1.1.41).

follows that $\delta(\text{ext}_{\psi_*}\theta) = 0$ because

$$(\text{ext}_{\psi_*}\theta)(q) = \begin{cases} (\psi'_*\theta)(q) & \text{if } q \in \psi(M), \\ 0 & \text{if } q \in N \setminus \psi(M) \end{cases}$$

and $\delta \circ \psi'_* = \psi'_* \circ \delta$. This implies that $[e_N(\text{ext}_{\psi_*}\theta)]_N$ is indeed an element of W . Suppose now that also $\zeta \in \Omega_{0,\delta}^1 M$ is such that $e_M\zeta$ is a representative of $[A]_M$. Then we also have $[e_N(\text{ext}_{\psi_*}\zeta)]_N$, and we must prove that this is equal to $[e_N(\text{ext}_{\psi_*}\theta)]_N$ for ξ to be well defined. We know that $e_M\theta$ and $e_M\zeta$ are gauge equivalent, i.e. $d(e_M\theta - e_M\zeta) = 0$. Exploiting Lemma 2.3.5, we deduce that $d(\theta - \zeta)$ falls in the kernel of $e_M : \Omega_0^2 M \rightarrow \Omega^2 M$, which is the causal propagator for the normally hyperbolic operator \square_2 . Applying Proposition 1.3.20 to $e_M : \Omega_0^2 M \rightarrow \Omega^2 M$, we find $\eta \in \Omega_0^2 M$ such that $\square_2\eta = d(\theta - \zeta)$. We have already seen one of the advantages of dealing with push-forwards of isometric embeddings, that is $\delta \circ \psi'_* = \psi'_* \circ \delta$. Besides this there are also the identity $d \circ \psi'_* = \psi'_* \circ d$ and, above all, the possibility to give sense to ext_{ψ_*} also for k -forms with $k \neq 1$. From these observations it follows that

$$d(\text{ext}_{\psi_*}(\theta - \zeta)) = \text{ext}_{\psi_*}(d(\theta - \zeta)) = \text{ext}_{\psi_*}(\square_2\eta) = \square_2(\text{ext}_{\psi_*}\eta).$$

Exploiting Lemma 2.3.5, we deduce that

$$d(e_N(\text{ext}_{\psi_*}(\theta - \zeta))) = e_N(d(\text{ext}_{\psi_*}(\theta - \zeta))) = e_N(\square_2(\text{ext}_{\psi_*}\eta)) = 0$$

because $\text{ext}_{\psi_*}\eta$ has compact support as η .

Linearity is a direct consequence of the definition of ξ . We focus on the compatibility with the symplectic forms σ and ω . To this end we consider $[A]_M$ and $[B]_M$ in V . Then we find θ and ζ in $\Omega_{0,\delta}^1 M$ such that $e_M\theta \in [A]_M$ and $e_M\zeta \in [B]_M$. We are ready to evaluate $\omega(\xi[A]_M, \xi[B]_M)$:

$$\begin{aligned} \omega(\xi[A]_M, \xi[B]_M) &= \omega([e_N(\text{ext}_{\psi_*}\theta)]_N, [e_N(\text{ext}_{\psi_*}\zeta)]_N) \\ &= (e_N(\text{ext}_{\psi_*}\theta), \text{ext}_{\psi_*}\zeta)_{h,1} \\ &= ((\text{res}_{\psi_*} \circ e_N \circ \text{ext}_{\psi_*})\theta, \zeta)_{1,g} \\ &= (e_M\theta, \zeta)_{1,g} \\ &= \sigma([A]_M, [B]_M), \end{aligned}$$

where we used also the identity $\text{res}_{\psi_*} \circ e_N \circ \text{ext}_{\psi_*} = e_M$ (cfr. Lemma 2.2.4). \square

Now we have the second part of our candidate covariant functor. For each pair of objects $(\mathcal{M}, \Lambda^1 M, \delta d)$ and $(\mathcal{N}, \Lambda^1 N, \delta d)$ of \mathbf{ghs}^{EM} there exists a map

$$\begin{aligned} \mathcal{B} : \mathbf{Mor}_{\mathbf{ghs}^{EM}}((\mathcal{M}, \Lambda^1 M, \delta d), (\mathcal{N}, \Lambda^1 N, \delta d)) &\rightarrow \mathbf{Mor}_{\mathbf{ssp}}((V, \sigma), (W, \omega)) \\ (\psi, \psi_*) &\mapsto \xi \end{aligned}$$

defined in accordance with our last lemma, where (V, σ) and (W, ω) respectively denote the symplectic spaces $\mathcal{B}(\mathcal{M}, \Lambda^1 M, \delta d)$ and $\mathcal{B}(\mathcal{N}, \Lambda^1 N, \delta d)$. To complete the classical theory of the electromagnetic field, it remains only to check that \mathcal{B} is actually a covariant functor. The next theorem answers to this question and provides also the causality property and the time slice axiom for \mathcal{B} .

Theorem 2.3.17. *Consider the map*

$$\begin{aligned} \mathcal{B} : \quad \text{Obj}_{\mathbf{ghs}^{EM}} &\rightarrow \text{Obj}_{\mathbf{ghs}^{EM}} \\ (\mathcal{M}, \Lambda^1 M, \delta d) &\mapsto (V, \sigma) \end{aligned}$$

defined in accordance with Lemma 2.3.13 and Lemma 2.3.14 and for each pair of objects $(\mathcal{M}, \Lambda^1 M, \delta d)$, $(\mathcal{N}, \Lambda^1 N, \delta d)$ of \mathbf{ghs}^{EM} consider the map

$$\begin{aligned} \mathcal{B} : \text{Mor}_{\mathbf{ghs}^{EM}}((\mathcal{M}, \Lambda^1 M, \delta d), (\mathcal{N}, \Lambda^1 N, \delta d)) &\rightarrow \text{Mor}_{\mathbf{ssp}}((V, \sigma), (W, \omega)) \\ (\psi, \psi_*) &\mapsto \xi \end{aligned}$$

defined in accordance with Lemma 2.3.16, where (V, σ) and (W, ω) respectively denote the symplectic spaces $\mathcal{B}(\mathcal{M}, \Lambda^1 M, \delta d)$ and $\mathcal{B}(\mathcal{N}, \Lambda^1 N, \delta d)$. These maps give rise to a covariant functor \mathcal{B} from the category \mathbf{ghs}^{EM} to the category \mathbf{ssp} . Moreover \mathcal{B} possesses the following properties:

- *causality: for each $(\mathcal{M}_1, \Lambda^1 M_1, \delta d)$, $(\mathcal{M}_2, \Lambda^1 M_2, \delta d)$, $(\mathcal{M}, \Lambda^1 M, \delta d)$ in \mathbf{ghs}^{EM} , each morphism (ψ_1, ψ_{1*}) from $(\mathcal{M}_1, \Lambda^1 M_1, \delta d)$ to $(\mathcal{M}, \Lambda^1 M, \delta d)$ and each morphism (ψ_2, ψ_{2*}) from $(\mathcal{M}_2, \Lambda^1 M_2, \delta d)$ to $(\mathcal{M}, \Lambda^1 M, \delta d)$ such that $\psi_1(M_1)$ and $\psi_2(M_2)$ are \mathcal{M} -causally separated subsets of M , it holds that*

$$\sigma(\xi_1[A_1]_{M_1}, \xi_2[A_2]_{M_2}) = 0$$

for each $[A_1]_{M_1} \in V_1$ and each $[A_2]_{M_2} \in V_2$, where (V_1, σ_1) , (V_2, σ_2) and (V, σ) are the symplectic spaces obtained with the application of \mathcal{B} respectively to $(\mathcal{M}_1, \Lambda^1 M_1, \delta d)$, $(\mathcal{M}_2, \Lambda^1 M_2, \delta d)$ and $(\mathcal{M}, \Lambda^1 M, \delta d)$, while $\xi_1 = \mathcal{B}(\psi_1, \psi_{1*})$ and $\xi_2 = \mathcal{B}(\psi_2, \psi_{2*})$;

- *time slice axiom: for each $(\mathcal{M}, \Lambda^1 M, \delta d)$ and $(\mathcal{N}, \Lambda^1 N, \delta d)$ in $\text{Obj}_{\mathbf{ghs}^{EM}}$ and each morphism (ψ, ψ_*) from $(\mathcal{M}, \Lambda^1 M, \delta d)$ to $(\mathcal{N}, \Lambda^1 N, \delta d)$ such that $\psi(M)$ includes a smooth spacelike Cauchy surface Σ for \mathcal{N} , it holds that*

$$\xi(V) = W,$$

where (V, σ) and (W, ω) are the symplectic spaces obtained with the application of \mathcal{B} respectively to $(\mathcal{M}, \Lambda^1 M, \delta d)$ and $(\mathcal{N}, \Lambda^1 N, \delta d)$, while $\xi = \mathcal{B}(\psi, \psi_*)$. In particular ξ is bijective and its inverse ξ^{-1} is a morphism of \mathbf{ssp} from (W, ω) to (V, σ) .

Proof. Whenever it is possible, this proof imitates that of Theorem 2.2.6, the main difference being due to the presence of the equivalence classes. Once that this fact is kept in mind, the verification of the covariant axioms and of the causality property is identical.

We must still check the time slice axiom. Since W is codomain of ξ , the inclusion $\xi(V) \subseteq W$ is trivial and we must prove the converse inclusion to complete the proof. To this end consider $[A]_N \in W$. We look for a section $\theta \in \Omega_{0,\delta}^1 M$ such that $e_N(\text{ext}_{\psi_*} \theta) \in [A]_N$. We observe that $[A]_N$ has a representative of the form $e_N \zeta$ for $\zeta \in \Omega_{0,\delta}^1 N$ that we denote with A , hence, exploiting the support properties of the Green operators and Proposition 1.2.18, we deduce that $\text{supp}(A) \cap \Sigma$ is a compact subset of Σ . Then we start with the usual procedure (refer to the proof of Theorem 2.2.6) applied to the normally hyperbolic operator \square_1 . Remember that now we have $\square_1 A = 0$, but also $\delta A = 0$ because we can exploit Lemma 2.3.5.

This entails that we find a decomposition $A = A^+ + A^-$, where A^\pm has \mathcal{M} -past/future compact support. Moreover we have that $\square_1 A^+ = -\square_1 A^-$ and $\delta A^+ = -\delta A^-$ are elements of $\Omega_0^1 N$ with support included in $\psi(M)$. We use them to define an element of $\Omega_0^1 M$ via restriction:

$$\theta = \text{res}_{\psi_*} (\delta dA^+) = \text{res}_{\psi_*} (\square_1 A^+ - d\delta A^+).$$

Trivially $\delta\theta = 0$ because $\delta \circ \psi'_* = \psi'_* \circ \delta$ so that $\theta \in \Omega_{0,\delta}^1 M$. Now we check that θ is exactly the one we were looking for. First of all θ has compact support so that $\text{ext}_{\psi_*} \theta$ has compact support too and hence we can apply $e_N^{a/r}$ to it obtaining

$$\begin{aligned} e_N^a(\text{ext}_{\psi_*} \theta) &= +e_N^a(\square_1 A^+ - d\delta A^+), \\ e_N^r(\text{ext}_{\psi_*} \theta) &= -e_N^r(\square_1 A^- - d\delta A^-). \end{aligned}$$

Now we exploit the \mathcal{M} -past/future compact support of A^\pm to apply Lemma 1.3.17. Furthermore we bear in mind that $\delta A^+ = -\delta A^-$ has compact support so that we can apply also Lemma 2.3.5. In this way we find

$$\begin{aligned} e_N^a(\text{ext}_{\psi_*} \theta) &= +A^+ - d(e_N^a(\delta A^+)), \\ e_N^r(\text{ext}_{\psi_*} \theta) &= -A^- + d(e_N^r(\delta A^-)). \end{aligned}$$

The last two equations together give

$$e_N(\text{ext}_{\psi_*} \theta) = A - d(e_N^a(\delta A^+) + e_N^r(\delta A^-)) = e_N \zeta - d(e_N(\delta A^+)).$$

This completes the proof because

$$d(e_N(\text{ext}_{\psi_*} \theta) - e_N \zeta) = -d(d(e_N(\delta A^+))) = 0,$$

hence

$$\xi [e_M \theta]_M = [e_N (\text{ext}_{\psi_*} \theta)]_N = [e_N \zeta]_N = [\mathbf{A}]_N$$

and in particular we deduce that $[\mathbf{A}]_N \in \xi(V)$. For the freedom in the choice of $[\mathbf{A}]_N \in W$, this fact implies the inclusion $W \subseteq \xi(V)$. The last part of the statement of the time slice axiom follows directly because each symplectic map is automatically injective (cfr. Remark 1.4.10) and the time slice axiom assures that ξ is also surjective, hence the inverse ξ^{-1} exists and it is trivial to check that it is a symplectic map too. \square

Now that we have the covariant functor \mathcal{B} describing the classical theory of the electromagnetic field and we know that it satisfies both the causality condition and the slice axiom. We can proceed with the quantization procedure composing \mathcal{B} with the covariant functor $\mathcal{C} : \mathbf{ssp} \rightarrow \mathbf{alg}$ defined in Lemma 2.2.7. In this way we obtain a locally covariant quantum field theory $\mathcal{A} = \mathcal{C} \circ \mathcal{B} : \mathbf{ghs}^{EM} \rightarrow \mathbf{alg}$ for the electromagnetic field which is causal and fulfils the time slice axiom (cfr. Theorem 2.2.8). Then Theorem 2.1.9, together with Remark 2.2.9, entails that, on each globally hyperbolic spacetime \mathcal{M} , \mathcal{A} provides the quantum field theory of the electromagnetic field $\mathcal{A}(\mathcal{M})$ in accordance with the Haag-Kastler axioms.

Chapter 3

Relative Cauchy evolution

The current chapter is devoted to the presentation of the *relative Cauchy evolution* (in the following often indicated by the acronym *RCE*) as it has been recently defined in [FV11]. We give a sketch of the idea: Suppose that a locally covariant quantum field theory \mathcal{A} fulfilling the time slice axiom is given (if the time slice axiom does not hold, we cannot define the RCE at all). The assignment of a globally hyperbolic spacetime \mathcal{M} induces via \mathcal{A} the assignment of a unital C*-algebra $\mathcal{A}(\mathcal{M})$ (cfr. Definition 2.1.5). Consider now another globally hyperbolic spacetime \mathcal{M}' with the same underlying manifold that coincides with \mathcal{M} outside a compact subset in which the metric of \mathcal{M}' is a perturbation (in a proper sense) of the metric of \mathcal{M} . Then also on \mathcal{M}' we have a unital C*-algebra $\mathcal{A}(\mathcal{M}')$. The RCE establishes the relation between the perturbed unital C*-algebra $\mathcal{A}(\mathcal{M}')$ and the original unital C*-algebra $\mathcal{A}(\mathcal{M})$.

If the LCQFT \mathcal{A} we are dealing with satisfies also the causality condition, as it was shown in Theorem 2.1.9, via \mathcal{A} we can obtain on each globally hyperbolic spacetime a quantum field theory according to the axiomatic approach proposed by Haag and Kastler in [HK64]. Therefore, when \mathcal{A} is also causal, we may interpret the RCE as a relation between the perturbed quantum field theory $\mathcal{A}(\mathcal{M}')$ and the original quantum field theory $\mathcal{A}(\mathcal{M})$, namely it tells us how the observables over \mathcal{M} are transformed when we change \mathcal{M} into \mathcal{M}' and then we go back to \mathcal{M} , i.e. when we perform a fluctuation of the spacetime metric.

We conclude that we have at our disposal an instrument that makes it possible to study the effects of fluctuations of the underlying metric on the quantum theory of some field for which we are able to construct a LCQFT fulfilling both the causality condition and the time slice axiom (the causality condition being required only to give sense to the interpretation in terms of observables, but not really indispensable for the definition of the RCE). The importance of this tool relies in the subsequent considerations. Till this point we dealt with quantum field theories on fixed globally hyperbolic spacetimes. However we know that the spacetime where we live is a solution of the Einstein's equation, hence it depends on the energy-matter content

of the whole universe. Indeed if we have a quantum field, we also expect to have its contribution to the stress-energy tensor appearing on the RHS of the Einstein equation and we may try to account for this contribution adding the expectation value of the stress-energy tensor associated to the quantum field. In this way the so-called semiclassical Einstein's equation arise (for a detailed discussion on this topic refer to [Wal94, Sect. 4.6, p. 85]). What we expect from such equation is a back-reaction effect: Quantum fields contribute to the stress-energy tensor which affects the solution of the semiclassical Einstein's equation, hence the spacetime metric, giving rise to a sort of perturbation of the quantum field itself.

When we are looking for solutions of the semiclassical Einstein's equation in the presence of a quantum field, we cannot forget of this back-reaction effect. Our aim is to show that the RCE is the proper tool to account for this effect when we deal with the Klein-Gordon field, the Proca field or the electromagnetic field. This fact was originally conjectured by Brunetti, Fredenhagen and Verch in [BFV03]: They supposed that the action of the functional derivative of the RCE with respect to the spacetime metric agrees with the action of the quantized stress-energy tensor and they showed that in any case the functional derivative of the RCE is symmetric and divergence free (both these properties are required to hold for any stress-energy tensor to be consistent with the LHS of the Einstein's equation). Moreover they verified their conjecture in the case of the Klein-Gordon field.

In the first part of this chapter we define the RCE following [FV11]. Although it is equivalent to the definition originally proposed in [BFV03] (for the proof of the equivalence refer to [FV11]), this approach seems to be more practical in some respects. Then we define the functional derivative of the RCE with respect to the spacetime metric (with reference to [BFV03]) and we show that this object is symmetric and divergence free. In the second part we deal with the Klein-Gordon field, the Proca field and the electromagnetic field. In first place we present the relation between the functional derivative of the RCE and the stress-energy tensor found by Brunetti, Fredenhagen and Verch in the case of the Klein-Gordon field and in second place we show that similar results hold also for the Proca field and for the electromagnetic field. In this way it is proved that the action of the functional derivative of the RCE agrees with the action of the quantized stress-energy tensor not only for the Klein-Gordon field, but also in the cases of the Proca field and of the electromagnetic field, thus confirming the conjecture that the functional derivative of the RCE behaves like the quantized stress-energy tensor associated to the field.

3.1 Definition and some properties

3.1.1 Procedure to define the relative Cauchy evolution

Following [FV11], we assume that a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ is given and we consider a compactly supported section $h \in \mathcal{D}(M, T^*M \otimes_s T^*M)$ in the symmetric tensor product of T^*M with itself. Then $g_h = g + h$ is indeed a section in $T^*M \otimes_s T^*M$ (cfr. Remark 1.1.26). If we assume that h is such that g_h is a Lorentzian metric, then (M, g_h) is a Lorentzian manifold. We can also require that h is such that (M, g_h) is time orientable. Since g_h coincides with g outside $\text{supp}(h)$, there exists only one connected component of the set of everywhere g_h -timelike vector fields over M which includes an element that coincides with some element of \mathfrak{t} outside $\text{supp}(h)$, i.e. there exists only one time orientation \mathfrak{t}_h for the time orientable Lorentzian manifold (M, g_h) that agrees with \mathfrak{t} outside the support of h . In this way we obtain the oriented and time oriented Lorentzian manifold $(M, g_h, \mathfrak{o}, \mathfrak{t}_h)$.

Definition 3.1.1. Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be a globally hyperbolic spacetime. $h \in \mathcal{D}(M, T^*M \otimes_s T^*M)$ is said to be an \mathcal{M} -globally hyperbolic perturbation of the metric g if the oriented and time oriented Lorentzian manifold $(M, g_h, \mathfrak{o}, \mathfrak{t})$ built above is actually a globally hyperbolic spacetime. In this case we denote the globally hyperbolic spacetime generated by the perturbation with $\mathcal{M}[h] = (M, g_h, \mathfrak{o}, \mathfrak{t}_h)$.

We denote the set of the \mathcal{M} -globally hyperbolic perturbations of the metric g with $GHP(\mathcal{M})$ and we endow such set with the topology induced by the usual topology of $\mathcal{D}(M, T^*M \otimes_s T^*M)$. Moreover for each compact subset K of M we define the subset $GHP(\mathcal{M}, K)$ of the \mathcal{M} -globally hyperbolic perturbations of the metric g with support contained in K .

Note that for each $\mathcal{M} \in \text{Obj}_{\text{ghs}}$ the set $GHP(\mathcal{M})$ is not empty because it contains at least the null section in $T^*M \otimes_s T^*M$. As a matter of fact one can show that for each $\mathcal{M} \in \text{Obj}_{\text{ghs}}$ there exists a neighborhood of the null section in $C^\infty(M, T^*M \otimes_s T^*M)$ which is included in $GHP(\mathcal{M})$. A similar conclusion holds also for $GHP(\mathcal{M}, K)$ for each compact subset K of M .

Before we define the relative Cauchy evolution, a lemma showing that the upcoming definition makes sense is required. The statement holds choosing all the upper signs or, alternatively, all the lower signs when \pm and \mp appear.

Lemma 3.1.2. Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be a globally hyperbolic spacetime and consider a compact subset K of M . We set $M_\pm = M \setminus J_\mp^\mathcal{M}(K)$. Then the following conclusions hold true:

- for each $h \in GHP(\mathcal{M}, K)$, M_\pm is an \mathcal{M} -causally convex and $\mathcal{M}[h]$ -causally convex connected open subset of M and the globally hyperbolic spacetimes $\mathcal{M}|_{M_\pm}$ and $\mathcal{M}[h]|_{M_\pm}$ coincide;

- *there exists a smooth spacelike Cauchy surface Σ_{\pm} for \mathcal{M} contained in M_{\pm} which is also a smooth spacelike Cauchy surface for $\mathcal{M}[h]$ for each $h \in GHP(\mathcal{M}, K)$;*
- *the inclusion map $\iota_{M_{\pm}}^M : M_{\pm} \rightarrow M$ can be seen as a morphism of \mathbf{ghs} from $\mathcal{M}|_{M_{\pm}} = \mathcal{M}[h]|_{M_{\pm}}$ to \mathcal{M} and as a morphism of \mathbf{ghs} from $\mathcal{M}[h]|_{M_{\pm}} = \mathcal{M}|_{M_{\pm}}$ to $\mathcal{M}[h]$ and its image includes a smooth spacelike Cauchy surface Σ_{\pm} for both \mathcal{M} and $\mathcal{M}[h]$.*

Proof. We focus on M_+ (the other case being similar). First of all we note that $J_-^{\mathcal{M}}(K)$ is a closed subset of M (see [BGP07, Lem. A.5.1, p. 173]) and hence M_+ is open.

Now we show that M_+ is \mathcal{M} -causally convex. By contradiction suppose that there exists a \mathbf{t} -future directed g -causal curve γ starting at $p \in M_+$ and ending at $q \in M_+$ which is not entirely contained in M_+ . Then we find a point r along γ that falls in $J_-^{\mathcal{M}}(K)$. It follows directly that p is a point of $J_-^{\mathcal{M}}(K)$ in contrast with the hypothesis that $p \in M_+$.

We fix $h \in GHP(\mathcal{M}, K)$ and we show that M_+ is also $\mathcal{M}[h]$ -causally convex. Again by contradiction suppose that there exists a \mathbf{t}_h -future directed g_h -causal curve γ starting at $p \in M_+$ and ending at $q \in M_+$ which is not entirely contained in M_+ . We deduce that γ intersects $J_-^{\mathcal{M}}(K)$. We consider the piece γ' of γ starting from p and ending in a point r of the boundary of $J_-^{\mathcal{M}}(K)$ so that γ' is outside $J_-^{\mathcal{M}}(K)$, except for the point r (which falls in $J_-^{\mathcal{M}}(K)$ because it is closed). Since $\text{supp}(h) \subseteq K$ and \mathbf{t}_h agrees with \mathbf{t} outside $\text{supp}(h)$, we conclude that γ' is also a \mathbf{t} -future directed g -causal curve from p to $r \in J_-^{\mathcal{M}}(K)$. Then we deduce that $p \in J_-^{\mathcal{M}}(K)$ in contrast with the hypothesis $p \in M_+$.

To prove connectedness, we apply Theorem 1.2.15 to \mathcal{M} . In this way we find a smooth spacelike Cauchy surface Σ for \mathcal{M} and a diffeomorphism $\psi : M \rightarrow \mathbb{R} \times \Sigma$. Then we define $\tau = \text{pr}_1 \circ \psi : M \rightarrow \mathbb{R}$, where pr_1 denotes the projection upon the first argument of the Cartesian product $\mathbb{R} \times \Sigma$. We realize immediately that τ is continuous (actually smooth). Since K is compact, we find $t \in \mathbb{R}$ such that $t > \sup \{\tau(p) : p \in K\}$. Then we consider $\Sigma_t = \psi^{-1}(\{t\} \times \Sigma)$. This is a smooth spacelike Cauchy surface for \mathcal{M} due to Theorem 1.2.15, in particular it is also connected. Moreover by construction $\Sigma_t \subseteq M_+$. Take now two arbitrary points p and q in M_+ and consider two inextensible \mathbf{t} -future directed g -timelike curves γ_p and γ_q such that γ_p passes through p and γ_q passes through q . These curves indeed meet Σ_t because it is a Cauchy surface. We denote with r and s the intersections of γ_p and respectively γ_q with Σ_t and we consider the pieces γ_{pr} and γ_{qs} of γ_p and respectively γ_q connecting p to r and q to s . From \mathcal{M} -causal convexity it follows that γ_{pr} and γ_{qs} are entirely contained in M_+ because also r and s fall in M_+ . Exploiting connectedness of Σ_t , we find γ_{rs} connecting r to s . Reversing γ_{qs} and pasting the result with γ_{pr} and γ_{rs} , we obtain a curve which connects p to q . This shows that

M_+ is connected.

Up to now we have shown that M_+ is an \mathcal{M} -causally convex and $\mathcal{M}[h]$ -causally convex connected open subset of M for each $h \in GHP(\mathcal{M}, K)$. Applying Proposition 1.2.16 and Remark 1.2.10, we deduce that $\mathcal{M}|_{M_+} = (M_+, g|_{M_+}, \mathfrak{o}|_{M_+}, \mathfrak{t}|_{M_+})$ and $\mathcal{M}[h]|_{M_+} = (M_+, g_h|_{M_+}, \mathfrak{o}|_{M_+}, \mathfrak{t}_h|_{M_+})$ are globally hyperbolic spacetimes for each $h \in GHP(\mathcal{M}, K)$. Now fix an arbitrary $h \in GHP(\mathcal{M}, K)$. Then $\text{supp}(h) \subseteq K$ and \mathfrak{t}_h agrees with \mathfrak{t} outside $\text{supp}(h)$. These facts entail that $g_h|_{M_+} = g|_{M_+}$ and that $\mathfrak{t}_h|_{M_+}$ and $\mathfrak{t}|_{M_+}$ induce the same time orientations on the Lorentzian manifolds $(M_+, g|_{M_+})$ and $(M_+, g_h|_{M_+})$, therefore we conclude $\mathcal{M}[h]|_{M_+} = \mathcal{M}|_{M_+}$ and the proof of the first point is complete.

As for the second point, we already determined a smooth spacelike Cauchy surface $\Sigma_t \subseteq M_+$ for \mathcal{M} . Now we show that this one is also a Cauchy surface for $\mathcal{M}[h]$ for each $h \in GHP(\mathcal{M}, K)$. Fix $h \in GHP(\mathcal{M}, K)$. Since g_h and g coincide on M_+ , Σ_t is spacelike also with respect to g_h . Consider an inextensible \mathfrak{t}_h -future directed g_h -timelike curve γ in M . There are two possibilities: If γ does not meet $\text{supp}(h)$, then it is also an inextensible \mathfrak{t} -future directed g -timelike curve in M and hence it must meet Σ_t exactly once; conversely if γ meets $\text{supp}(h)$, we can consider the piece γ' of γ that lies in $J_+^{\mathcal{M}}(K) \setminus K$. γ' is a \mathfrak{t} -future directed g -timelike curve in M which is by construction inextensible in the future. We can extend it in the past in such a way that the result is an inextensible \mathfrak{t} -future directed g -timelike curve γ'' in M . Then γ'' meets Σ_t exactly once. The choice of $t > \sup\{\tau(p) : p \in K\}$ entails that K lies in the \mathcal{M} -causal past of Σ_t and that $K \cap \Sigma_t = \emptyset$. Then the only intersection of γ'' with Σ_t must fall in the \mathcal{M} -causal future of K . We deduce that γ' already met Σ_t , and hence also γ . Note that the other piece of γ (the one not contained in $J_+^{\mathcal{M}}(K) \setminus K$) cannot intersect Σ_t because it is contained in $J_-^{\mathcal{M}}(K)$. Hence also in the second case γ meets Σ_t exactly once.

We turn our attention to the last point and we begin noting that $\iota_{M_+}^M$ is an embedding (cfr. Remark 1.1.7). Now fix $h \in GHP(\mathcal{M}, K)$. Undoubtedly $\iota_{M_+}^M$ is isometric and preserves both orientation and time orientation whether we consider \mathcal{M} or $\mathcal{M}[h]$ as target since $\iota_{M_+}^{M*}g = g|_{M_+} = g_h|_{M_+}$, $\iota_{M_+}^{M*}\mathfrak{o} = \mathfrak{o}|_{M_+}$ and $\iota_{M_+}^{M*}\mathfrak{t}_h = \mathfrak{t}_h|_{M_+}$ and $\iota_{M_+}^{M*}\mathfrak{t} = \mathfrak{t}|_{M_+}$ ¹ induce the same time orientations on the Lorentzian manifolds $(M_+, g|_{M_+})$ and $(M_+, g_h|_{M_+})$ (which are the same as a matter of fact). Hence $\iota_{M_+}^M$ is an isometric embedding which preserves both orientation and time orientation whether we consider \mathcal{M} or $\mathcal{M}[h]$ as target (we are considering $\mathcal{M}[h]|_{M_+} = \mathcal{M}|_{M_+}$ as source). The image of $\iota_{M_+}^M$ is trivially M_+ , which is causally convex with respect to both \mathcal{M} and $\mathcal{M}[h]$. Moreover we showed that Σ_t is a smooth spacelike Cauchy surface for both \mathcal{M} and $\mathcal{M}[h]$ that is contained in M_+ . These observations concludes

¹Note that pulling back \mathfrak{t} and \mathfrak{t}_h through $\iota_{M_+}^M$ means that we are taking any representative (which is a vector field) restricted to M_+ and we are pushing it forward through the diffeomorphism $\iota_{M_+}^{M'} : M_+ \rightarrow M_+$ induced by the embedding $\iota_{M_+}^M$.

the proof. \square

Consider a globally hyperbolic spacetime \mathcal{M} and take $h \in GHP(\mathcal{M})$. Applying the last lemma with $K = \text{supp}(h)$, we have the following diagrams:

$$\begin{array}{ccccc} \mathcal{M} & \xleftarrow{\iota_{M-}^M} & \mathcal{M}|_{M-} & = & \mathcal{M}[h]|_{M-} & \xrightarrow{\iota_{M-}^M} & \mathcal{M}[h], \\ \mathcal{M} & \xleftarrow{\iota_{M+}^M} & \mathcal{M}|_{M+} & = & \mathcal{M}[h]|_{M+} & \xrightarrow{\iota_{M+}^M} & \mathcal{M}[h]. \end{array}$$

Note that here the arrows represent morphisms of the category \mathbf{ghs} whose image includes a smooth spacelike Cauchy surface of the target object (namely a globally hyperbolic spacetime). We introduce a convenient notation rewriting the diagrams above (each element of the new diagrams is defined by the element of the old diagram which occupies the same position):

$$\begin{array}{ccccc} \mathcal{M} & \xleftarrow{\iota_-^{\mathcal{M}}[h]} & \mathcal{M}_-[h] & \xrightarrow{j_-^{\mathcal{M}}[h]} & \mathcal{M}[h], \\ \mathcal{M} & \xleftarrow{\iota_+^{\mathcal{M}}[h]} & \mathcal{M}_+[h] & \xrightarrow{j_+^{\mathcal{M}}[h]} & \mathcal{M}[h]. \end{array}$$

The main advantage of the new notation relies in the fact that we can recognize from the name if we are considering $\iota_{M\pm}^M$ as a morphism from $\mathcal{M}|_{M\pm}$ to \mathcal{M} (in which case we use the symbol ι) or as a morphism from $\mathcal{M}[h]|_{M\pm}$ to $\mathcal{M}[h]$ (in which case we use the symbol j). Moreover this notation emphasizes the dependence on h of all the elements actually depend in some way on the choice of h in $GHP(\mathcal{M})$.

If we consider a locally covariant quantum field theory \mathcal{A} , the diagrams above are mapped to

$$\begin{array}{ccccc} \mathcal{A}(\mathcal{M}) & \xleftarrow{\mathcal{A}(\iota_-^{\mathcal{M}}[h])} & \mathcal{A}(\mathcal{M}_-[h]) & \xrightarrow{\mathcal{A}(j_-^{\mathcal{M}}[h])} & \mathcal{A}(\mathcal{M}[h]), \\ \mathcal{A}(\mathcal{M}) & \xleftarrow{\mathcal{A}(\iota_+^{\mathcal{M}}[h])} & \mathcal{A}(\mathcal{M}_+[h]) & \xrightarrow{\mathcal{A}(j_+^{\mathcal{M}}[h])} & \mathcal{A}(\mathcal{M}[h]), \end{array}$$

where all the arrows now are morphisms of the category \mathbf{alg} . If we suppose that \mathcal{A} fulfils the time slice axiom, we deduce that all the morphisms are actually unit preserving *-isomorphisms between unital C*-algebras. This fact is a consequence of the time slice axiom, together with Lemma 3.1.2. Reversing the arrows on the left in the last two diagrams, we can define the following *-isomorphisms between unital C*-algebras:

$$\begin{aligned} \tau_-^{\mathcal{M}}[h] &= \mathcal{A}(j_-^{\mathcal{M}}[h]) \circ \mathcal{A}(\iota_-^{\mathcal{M}}[h])^{-1} : \mathcal{A}(\mathcal{M}) \rightarrow \mathcal{A}(\mathcal{M}[h]), \\ \tau_+^{\mathcal{M}}[h] &= \mathcal{A}(j_+^{\mathcal{M}}[h]) \circ \mathcal{A}(\iota_+^{\mathcal{M}}[h])^{-1} : \mathcal{A}(\mathcal{M}) \rightarrow \mathcal{A}(\mathcal{M}[h]). \end{aligned}$$

We are ready to define the relative Cauchy evolution.

Definition 3.1.3. Consider a LCQFT \mathcal{A} fulfilling the time slice axiom. For each globally hyperbolic spacetime \mathcal{M} and each $h \in GHP(\mathcal{M})$, we call *relative Cauchy*

evolution (or briefly RCE) induced by h on \mathcal{M} the following $*$ -automorphism of the unital C^* -algebra $\mathcal{A}(\mathcal{M})$:

$$R_h^{\mathcal{M}} = (\tau_-^{\mathcal{M}}[h])^{-1} \circ \tau_+^{\mathcal{M}}[h] : \mathcal{A}(\mathcal{M}) \rightarrow \mathcal{A}(\mathcal{M}).$$

Exploiting the expressions of $\tau_{\pm}^{\mathcal{M}}[h]$, we may rewrite the RCE in the following way:

$$R_h^{\mathcal{M}} = \mathcal{A}(i_-^{\mathcal{M}}[h]) \circ \mathcal{A}(j_-^{\mathcal{M}}[h])^{-1} \circ \mathcal{A}(j_+^{\mathcal{M}}[h]) \circ \mathcal{A}(i_+^{\mathcal{M}}[h])^{-1}. \quad (3.1.1)$$

As a consequence of the functorial properties of \mathcal{A} , we expect that the RCE on a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ is insensitive to changes in the fluctuations h of the spacetime metric g produced by an orientation preserving diffeomorphism from the oriented manifold (M, \mathfrak{o}) to itself that acts trivially outside of a compact subset of M including the support of h .

Proposition 3.1.4. *Let \mathcal{A} be a LCQFT fulfilling the time slice axiom, let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be a globally hyperbolic spacetime and let ψ be an orientation preserving diffeomorphism from (M, \mathfrak{o}) to itself acting trivially outside of a compact subset K of M . Consider $h \in GHP(\mathcal{M}, K)$ such that $h' = \psi_* g_h - g \in GHP(\mathcal{M}, K)$. Then the diffeomorphism $\psi : M \rightarrow M$ may be seen as an orientation and time orientation preserving isometric diffeomorphism from $\mathcal{M}[h]$ to $\mathcal{M}[h']$ and $R_h^{\mathcal{M}} = R_{h'}^{\mathcal{M}}$.*

Proof. Recall that $\mathcal{M}[h] = (M, g_h, \mathfrak{o}, \mathfrak{t}_h)$ and $\mathcal{M}[h'] = (M, g_{h'}, \mathfrak{o}, \mathfrak{t}_{h'})$. Exploiting the hypothesis, we deduce

$$\psi_* g_h = g + h' = g_{h'}.$$

From this fact it follows that $\mathfrak{t}_{h'}$ is one of the connected components of the set of everywhere $\psi_* g_h$ -timelike vector fields over M . Furthermore \mathfrak{t}_h is by definition one of the connected components of the set of everywhere g_h -timelike vector fields over M , hence $\psi_* \mathfrak{t}_h$ is one of the connected components of the set of everywhere $\psi_* g_h$ -timelike vector fields over M . \mathfrak{t}_h agrees with \mathfrak{t} outside $\text{supp}(h)$ by definition of $\mathcal{M}[h]$, while $\mathfrak{t}_{h'}$ agrees with \mathfrak{t} outside $\text{supp}(h')$ by definition of $\mathcal{M}[h']$, hence \mathfrak{t}_h and $\mathfrak{t}_{h'}$ agree outside K . Moreover $\psi_* \mathfrak{t}_h = \mathfrak{t}_h$ outside K because by hypothesis ψ acts trivially outside K . Then we conclude that $\psi_* \mathfrak{t}_h$ and $\mathfrak{t}_{h'}$ agree outside K , therefore they are the same connected component of the set of everywhere $\psi_* g_h$ -timelike vector fields over M , i.e. $\psi_* \mathfrak{t}_h = \mathfrak{t}_{h'}$. This shows that actually ψ may be interpreted as an orientation and time orientation isometric diffeomorphism from $\mathcal{M}[h]$ to $\mathcal{M}[h']$.

Now we focus on the second part of the statement. We begin defining $M_{\pm} = M \setminus J_{\mp}^{\mathcal{M}}(K)$. Since h and h' are elements of $GHP(\mathcal{M}, K)$, we can apply Lemma 3.1.2 to deduce that M_{\pm} is an \mathcal{M} -causally convex, $\mathcal{M}[h]$ -causally convex and $\mathcal{M}[h']$ -causally convex connected open subset of M containing a smooth spacelike Cauchy surface Σ_{\pm} for \mathcal{M} that is also a smooth spacelike Cauchy surface for both $\mathcal{M}[h]$

and $\mathcal{M}[h']$. Taking into account $\mathcal{M}_\pm[h]$ and $\mathcal{M}_\pm[h']$ (whose underlying manifolds are respectively $M \setminus J_\mp(\text{supp}(h))$ and $M \setminus J_\mp(\text{supp}(h'))$), we see that both include M_\pm and then also Σ_\pm . It turns out almost trivially that Σ_\pm is a smooth spacelike Cauchy surface for both $\mathcal{M}_\pm[h]$ and $\mathcal{M}_\pm[h']$. Hence M_\pm is also an $\mathcal{M}_\pm[h]$ -causally convex and $\mathcal{M}_\pm[h']$ -causally convex connected open subset containing a smooth spacelike Cauchy surface Σ_\pm for both $\mathcal{M}_\pm[h]$ and $\mathcal{M}_\pm[h']$. This fact entails that we can consider the globally hyperbolic spacetimes $\mathcal{M}_\pm[h]|_{M_\pm}$ and $\mathcal{M}_\pm[h']|_{M_\pm}$ and interpret the inclusion maps of M_\pm in $M \setminus J_\mp^\mathcal{M}(\text{supp}(h))$ and in $M \setminus J_\mp^\mathcal{M}(\text{supp}(h'))$ as morphisms of the category \mathbf{ghs} whose image includes a smooth spacelike Cauchy surface of the target:

$$\begin{aligned}\alpha_\pm &\in \text{Mor}_{\mathbf{ghs}}\left(\mathcal{M}_\pm[h]|_{M_\pm}, \mathcal{M}_\pm[h]\right), \\ \beta_\pm &\in \text{Mor}_{\mathbf{ghs}}\left(\mathcal{M}_\pm[h']|_{M_\pm}, \mathcal{M}_\pm[h']\right).\end{aligned}$$

We may also consider the globally hyperbolic spacetime $\mathcal{M}|_{M_\pm}$ and we realize that

$$\mathcal{M}_\pm[h]|_{M_\pm} = \mathcal{M}[h]|_{M_\pm} = \mathcal{M}|_{M_\pm} = \mathcal{M}[h']|_{M_\pm} = \mathcal{M}_\pm[h']|_{M_\pm}$$

because $g_h = g = g_{h'}$ outside K and \mathfrak{t}_h , \mathfrak{t} and $\mathfrak{t}_{h'}$ agree outside K . Hence we can consider α_\pm and β_\pm as morphisms starting from $\mathcal{M}|_{M_\pm}$. For convenience we recollect here the morphisms generated by the globally hyperbolic perturbations h and h' :

$$\begin{aligned}i_\pm^\mathcal{M}[h] &\in \text{Mor}_{\mathbf{ghs}}(\mathcal{M}_\pm[h], \mathcal{M}), \\ j_\pm^\mathcal{M}[h] &\in \text{Mor}_{\mathbf{ghs}}(\mathcal{M}_\pm[h], \mathcal{M}[h]), \\ i_\pm^\mathcal{M}[h'] &\in \text{Mor}_{\mathbf{ghs}}(\mathcal{M}_\pm[h'], \mathcal{M}), \\ j_\pm^\mathcal{M}[h'] &\in \text{Mor}_{\mathbf{ghs}}(\mathcal{M}_\pm[h'], \mathcal{M}[h']).\end{aligned}$$

We exploit α_\pm and β_\pm and the fact that their images include a smooth spacelike Cauchy surface of the target, together with the hypothesis that the time slice axiom holds for \mathcal{M} , to rewrite both $R_h^\mathcal{M}$ and $R_{h'}^\mathcal{M}$:

$$\begin{aligned}R_h^\mathcal{M} &= \mathcal{A}(i_-^\mathcal{M}[h]) \circ \mathcal{A}(j_-^\mathcal{M}[h])^{-1} \circ \mathcal{A}(j_+^\mathcal{M}[h]) \circ \mathcal{A}(i_+^\mathcal{M}[h])^{-1} \\ &= \mathcal{A}(i_-^\mathcal{M}[h]) \circ \mathcal{A}(\alpha_-) \circ \mathcal{A}(\alpha_-)^{-1} \circ \mathcal{A}(j_-^\mathcal{M}[h])^{-1} \\ &\quad \circ \mathcal{A}(j_+^\mathcal{M}[h]) \circ \mathcal{A}(\alpha_+) \circ \mathcal{A}(\alpha_+)^{-1} \circ \mathcal{A}(i_+^\mathcal{M}[h])^{-1} \\ &= \mathcal{A}(i_-^\mathcal{M}[h] \circ \alpha_-) \circ \mathcal{A}(j_-^\mathcal{M}[h] \circ \alpha_-)^{-1} \\ &\quad \circ \mathcal{A}(j_+^\mathcal{M}[h] \circ \alpha_+) \circ \mathcal{A}(i_+^\mathcal{M}[h] \circ \alpha_+)^{-1},\end{aligned}$$

$$\begin{aligned}
R_{h'}^{\mathcal{M}} &= \mathcal{A}(i_-^{\mathcal{M}}[h']) \circ \mathcal{A}(j_-^{\mathcal{M}}[h'])^{-1} \circ \mathcal{A}(j_+^{\mathcal{M}}[h']) \circ \mathcal{A}(i_+^{\mathcal{M}}[h'])^{-1} \\
&= \mathcal{A}(i_-^{\mathcal{M}}[h']) \circ \mathcal{A}(\beta_-) \circ \mathcal{A}(\beta_-)^{-1} \circ \mathcal{A}(j_-^{\mathcal{M}}[h'])^{-1} \\
&\quad \circ \mathcal{A}(j_+^{\mathcal{M}}[h']) \circ \mathcal{A}(\beta_+) \circ \mathcal{A}(\beta_+)^{-1} \circ \mathcal{A}(i_+^{\mathcal{M}}[h'])^{-1} \\
&= \mathcal{A}(i_-^{\mathcal{M}}[h'] \circ \beta_-) \circ \mathcal{A}(j_-^{\mathcal{M}}[h'] \circ \beta_-)^{-1} \\
&\quad \circ \mathcal{A}(j_+^{\mathcal{M}}[h'] \circ \beta_+) \circ \mathcal{A}(i_+^{\mathcal{M}}[h'] \circ \beta_+)^{-1}.
\end{aligned}$$

We can observe that $i_{\pm}^{\mathcal{M}}[h] \circ \alpha_{\pm}$ and $i_{\pm}^{\mathcal{M}}[h'] \circ \beta_{\pm}$ are both morphisms from $\mathcal{M}|_{M_{\pm}}$ to \mathcal{M} whose underlying map is nothing but the inclusion map of M_{\pm} into M , hence these morphisms are exactly the same and we denote both of them with ψ_{\pm} . Now we exploit the morphism ψ from $\mathcal{M}[h]$ to $\mathcal{M}[h']$:

$$\begin{aligned}
R_h^{\mathcal{M}} &= \mathcal{A}(\psi_-) \circ \mathcal{A}(j_-^{\mathcal{M}}[h] \circ \alpha_-)^{-1} \circ \mathcal{A}(j_+^{\mathcal{M}}[h] \circ \alpha_+) \circ \mathcal{A}(\psi_+)^{-1} \\
&= \mathcal{A}(\psi_-) \circ \mathcal{A}(j_-^{\mathcal{M}}[h] \circ \alpha_-)^{-1} \circ \mathcal{A}(\psi)^{-1} \\
&\quad \circ \mathcal{A}(\psi) \circ \mathcal{A}(j_+^{\mathcal{M}}[h] \circ \alpha_+) \circ \mathcal{A}(\psi_+)^{-1} \\
&= \mathcal{A}(\psi_-) \circ \mathcal{A}(\psi \circ j_-^{\mathcal{M}}[h] \circ \alpha_-)^{-1} \circ \mathcal{A}(\psi \circ j_+^{\mathcal{M}}[h] \circ \alpha_+) \circ \mathcal{A}(\psi_+)^{-1}.
\end{aligned}$$

We note that both $\psi \circ j_{\pm}^{\mathcal{M}}[h] \circ \alpha_{\pm}$ and $j_{\pm}^{\mathcal{M}}[h'] \circ \beta_{\pm}$ are morphisms from $\mathcal{M}|_{M_{\pm}}$ to $\mathcal{M}[h']$ and, since ψ acts trivially outside $K \subseteq M \setminus M_{\pm}$, we deduce that the underlying maps coincide. Hence $\psi \circ j_{\pm}^{\mathcal{M}}[h] \circ \alpha_{\pm}$ and $j_{\pm}^{\mathcal{M}}[h'] \circ \beta_{\pm}$ are actually the same morphism and we denote them with ψ^{\pm} . At this point we have

$$R_{h'}^{\mathcal{M}} = \mathcal{A}(\psi_-) \circ \mathcal{A}(\psi^-)^{-1} \circ \mathcal{A}(\psi^+) \circ \mathcal{A}(\psi_+)^{-1} = R_h^{\mathcal{M}}$$

and this concludes the proof. \square

3.1.2 Functional derivative of the relative Cauchy evolution

In this subsection we define the functional derivative of the relative Cauchy evolution with respect to the spacetime metric following the procedure presented in [BFV03] (adapted to the current definition of the RCE).

We consider a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$. For each $h \in GHP(\mathcal{M})$ we know that $\mathcal{M}[h] = (M, g_h, \mathfrak{o}, \mathfrak{t}_h)$ is a globally hyperbolic spacetime in its own right. Moreover, for each $h \in GHP(\mathcal{M})$, taking $M_{\pm} = M \setminus J_{\mp}^{\mathcal{M}}(\text{supp}(h))$ and applying Lemma 3.1.2 with $K = \text{supp}(h)$, we find that $\mathcal{M}_{\pm}[h] = \mathcal{M}|_{M_{\pm}}$ is a globally hyperbolic spacetimes including a smooth spacelike Cauchy surface for both \mathcal{M} and $\mathcal{M}[h]$.

We consider a locally covariant quantum field theory \mathcal{A} fulfilling the time slice axiom and we take into account the unital C*-algebra $\mathcal{A}(\mathcal{M})$.

Assumption 3.1.5. *Suppose that π is a representation of $\mathcal{A}(\mathcal{M})$ on a Hilbert space \mathcal{H} . Assume that there exist a dense subspace \mathcal{V} of \mathcal{H} and a dense unital*

sub- \ast -algebra \mathcal{B} of $\mathcal{A}(\mathcal{M})$ such that for each $\Omega \in \mathcal{V}$ and each $b \in \mathcal{B}$ the following conditions are satisfied:

- for each compact subset K of M and each smooth 1-parameter family

$$\begin{aligned} (-1, 1) &\rightarrow \text{GHP}(\mathcal{M}, K) \\ s &\mapsto h^s \end{aligned}$$

such that $h^0 = 0$, the map

$$\begin{aligned} (-1, 1) &\rightarrow \mathbb{C} \\ s &\mapsto \langle \Omega, \pi(R_{h^s}^{\mathcal{M}} b) \Omega \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of \mathcal{H} , is continuously differentiable;

- there exists a section $\beta \in C^\infty(M, TM \otimes_s TM)$ such that, for each compact subset K of M and each smooth 1-parameter family

$$\begin{aligned} (-1, 1) &\rightarrow \text{GHP}(\mathcal{M}, K) \\ s &\mapsto h^s \end{aligned}$$

verifying $h^0 = 0$, it holds that

$$\int_M \left(\frac{dh^s}{ds} \Big|_0 \right) (\beta) d\mu_g = \frac{d}{ds} \langle \Omega, \pi(R_{h^s}^{\mathcal{M}} b) \Omega \rangle \Big|_0, \quad (3.1.2)$$

where the dual pairing between $T^*M \otimes_s T^*M$ and $TM \otimes_s TM$ is taken into account and $d\mu_g$ is the standard volume form on \mathcal{M} .

Remark 3.1.6. Some remarks about the last assumption are required. First of all we explain the meaning of the integrand appearing on the LHS of eq. (3.1.2). Fix a compact subset K of M and consider a smooth 1-parameter family $s \mapsto h^s$ of the type required above. Using local coordinates at a point $p \in M$, we have the following expression for the components of $dh^s/ds|_0$ evaluated at p :

$$\left(\frac{dh^s}{ds} \Big|_0 (p) \right)_{ij} = \frac{d}{ds} h_{ij}^s(p) \Big|_0,$$

where $h_{ij}^s(p)$ are the components of h^s evaluated at p for some $s \in (-1, 1)$. From the assumption that $s \mapsto h^s$ is smooth, it follows that $dh^s/ds|_0$ is a section in $T^*M \otimes_s T^*M$. Since $\text{supp}(h^s)$ is contained in K for each $s \in (-1, 1)$, we deduce also that $dh^s/ds|_0$ has support included in K , hence compact. This fact assures that the integral makes sense.

Secondly we consider the term that appears on the RHS. The derivative appearing here is well defined as a direct consequence of the first point in the assumption

above.

Now that we have understood the meaning of both the LHS and the RHS of eq. (3.1.2), we can try to understand the consequences of this equation on the section β (which is supposed to exist). Fix a compact subset K of M . The freedom in the choice of the family $s \mapsto h^s$, together with the fact that $GHP(\mathcal{M}, K)$ includes a neighborhood of the null section in $C^\infty(M, T^*M \otimes_s T^*M)$, entails that β is uniquely determined on K : If we suppose that there exists another section β' of the same type satisfying the same equation, we deduce that

$$\int_M f(\beta - \beta') d\mu_g = 0 \quad \forall f \in \mathcal{D}(M, T^*M \otimes_s T^*M) : \text{supp}(f) \subseteq K$$

and, working with sections f with support contained in open subsets of M included in K on which the vector bundle $T^*M \otimes_s T^*M$ is trivialized, we conclude that $\beta - \beta' = 0$ on K due to the density of the vector space $\mathcal{D}(O, O \times \mathbb{R}^n)$ in the Banach space $L^p(O, O \times \mathbb{R}^n)$ for each open subset O of \mathbb{R}^d , each $d, n \in \mathbb{N}$ and each $p \in [1, \infty)$. Then the freedom in the choice of K entails that β is uniquely determined everywhere on M .

These observations entail that the assumption made above assures the uniqueness of the functional derivative β of $\langle \Omega, \pi(R_h^{\mathcal{M}} b) \Omega \rangle$ with respect to \mathcal{M} -globally hyperbolic perturbations of the spacetime metric. For brevity we will simply say that β is the functional derivative of $\langle \Omega, \pi(R_h^{\mathcal{M}} b) \Omega \rangle$ with respect to the spacetime metric and we will write $\frac{\delta}{\delta h} \langle \Omega, \pi(R_h^{\mathcal{M}} b) \Omega \rangle$ in place of β .

In the last remark we saw how Assumption 3.1.5 implies that $\frac{\delta}{\delta h} \langle \Omega, \pi(R_h^{\mathcal{M}} b) \Omega \rangle$ is uniquely defined for each Ω in a dense subspace \mathcal{V} of a proper Hilbert space \mathcal{H} and for each b in a proper dense sub- $*$ -algebra of $\mathcal{A}(\mathcal{M})$. We are ready to define the functional derivative of the RCE with respect to the spacetime metric.

Definition 3.1.7. Let \mathcal{A} be a LCQFT fulfilling the time slice axiom and let \mathcal{M} be a globally hyperbolic spacetime. Consider a representation π of $\mathcal{A}(\mathcal{M})$ on a Hilbert space \mathcal{H} . If Assumption 3.1.5 holds, there exist a dense subspace \mathcal{V} of \mathcal{H} and a dense unital sub- $*$ -algebra \mathcal{B} of $\mathcal{A}(\mathcal{M})$ such that we can uniquely define for each $b \in \mathcal{B}$ the *functional derivative with respect to the spacetime metric of the relative Cauchy evolution acting on b* (briefly *functional derivative of the RCE*), denoted by $\frac{\delta}{\delta h} \pi(R_h^{\mathcal{M}} b)$, as a quadratic form on \mathcal{V} :

$$\left\langle \Omega, \left(\frac{\delta}{\delta h} \pi(R_h^{\mathcal{M}} b) \right) \Omega \right\rangle = \frac{\delta}{\delta h} \langle \Omega, \pi(R_h^{\mathcal{M}} b) \Omega \rangle \quad \forall \Omega \in \mathcal{V}.$$

In [BFV03] it was conjectured that the action of the functional derivative of the RCE with respect to the spacetime metric agrees with the action of the quantized-stress energy tensor. The first properties to be checked in order to support such hypothesis are the symmetry and the null divergence. Per definition $\frac{\delta}{\delta h} \langle \Omega, \pi(R_h^{\mathcal{M}} b) \Omega \rangle$

is an element of $C^\infty(M, TM \otimes_s TM)$ for each $b \in \mathcal{B}$ and each $\Omega \in \mathcal{V}$, hence $\frac{\delta}{\delta h} \pi(R_h^{\mathcal{M}} b)$ is symmetric for each $b \in \mathcal{B}$ (in the sense of the quadratic forms on \mathcal{V}). The evaluation of the divergence is addressed in the following proposition.

First we need to introduce some notation previously. Since $\frac{\delta}{\delta h} \langle \Omega, \pi(R_h^{\mathcal{M}} b) \Omega \rangle$ is an element of $C^\infty(M, TM \otimes_s TM)$, at each point p of M we may write it in local coordinates. We denote its components at p with

$$\frac{\delta}{\delta h_{ij}(p)} \langle \Omega, \pi(R_h^{\mathcal{M}} b) \Omega \rangle.$$

Note that the indices are “doubly” covariant, hence contravariant, in accordance with the fact that $\frac{\delta}{\delta h} \langle \Omega, \pi(R_h^{\mathcal{M}} b) \Omega \rangle$ evaluated in a point $p \in M$ is an element of $T^{(2,0)}M$. Similarly we denote the components of $\frac{\delta}{\delta h} \pi(R_h^{\mathcal{M}} b)$ at p with

$$\frac{\delta}{\delta h_{ij}(p)} \pi(R_h^{\mathcal{M}} b).$$

Proposition 3.1.8. *Let \mathcal{A} be a LCQFT fulfilling the time slice axiom and let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ be a globally hyperbolic spacetime. Consider a representation π of $\mathcal{A}(\mathcal{M})$ on a Hilbert space \mathcal{H} such that Assumption 3.1.5 holds so that we find a dense subspace \mathcal{V} of \mathcal{H} and a dense unital sub- $*$ -algebra \mathcal{B} of $\mathcal{A}(\mathcal{M})$ on which the functional derivative of the RCE is defined. Then for each $b \in \mathcal{B}$ we have*

$$\nabla_i \left(\frac{\delta}{\delta h_{ij}(p)} \pi(R_h^{\mathcal{M}} b) \right) = 0 \quad \forall p \in M$$

in the sense of the quadratic forms on \mathcal{V} , where ∇ denotes the Levi-Civita connection with respect to the metric g .

Proof. The thesis is a formal expression meaning that for each $b \in \mathcal{B}$, each $\Omega \in \mathcal{V}$ and each $p \in M$

$$\nabla_i \left(\frac{\delta}{\delta h_{ij}(p)} \langle \Omega, \pi(R_h^{\mathcal{M}} b) \Omega \rangle \right) = 0.$$

We fix $b \in \mathcal{B}$ and $\Omega \in \mathcal{V}$ and, denoting $\frac{\delta}{\delta h} \langle \Omega, \pi(R_h^{\mathcal{M}} b) \Omega \rangle$ with β , we may rewrite the thesis in the following way: for each oriented local coordinate neighborhood (U, V, ϕ) and for each vector field $X \in \mathcal{D}(M, TM)$ with support included in U it holds

$$\int_V (\nabla_i \beta^{ij}) g_{jk} X^k \sqrt{|\det g|} dV = 0,$$

where dV denotes the standard volume form on \mathbb{R}^d and all sections in the integrand are meant in local coordinates. Via an integration by parts and since X is null on the boundary of U , we deduce that the last equation is equivalent to

$$\int_V \beta^{ij} (\nabla_i X_j) \sqrt{|\det g|} dV = 0.$$

We know that β is symmetric so that we can write

$$\beta^{ij} (\nabla_i X_j) = \frac{1}{2} (\beta^{ij} + \beta^{ji}) (\nabla_i X_j) = \frac{1}{2} \beta^{ij} (\nabla_i X_j + \nabla_j X_i),$$

hence our thesis finally becomes

$$\int_V \beta^{ij} (\nabla_i X_j + \nabla_j X_i) \sqrt{|\det g|} dV = 0 \quad (3.1.3)$$

for each local coordinate neighborhood (U, V, ϕ) and for each vector field $X \in \mathcal{D}(M, TM)$ with support included in U .

Fix now a local coordinate neighborhood (U, V, ϕ) and a compactly supported vector field $X \in \mathcal{D}(M, TM)$ with support K included in U . We know that each compactly supported vector field on M generates a 1-parameter group of diffeomorphisms of M $s \in \mathbb{R} \mapsto \psi^s$ acting trivially outside of K with $\psi^0 = \text{id}_M$ (cfr. [Jos95, Thm. 1.9.2, p. 49]). Note that ψ^s is necessarily orientation preserving for each $s \in \mathbb{R}$: Outside of K it acts trivially (hence its Jacobian determinant is positive); if it reverses some coordinate neighborhood inside K (i.e. its Jacobian determinant in that coordinate neighborhood is negative), then there exists a point in some coordinate neighborhood in which its Jacobian determinant is null, in contradiction with the fact that it is a diffeomorphism. Consider now $GHP(\mathcal{M}, K)$ and remember that it includes at least a neighborhood of the null section in $C^\infty(M, T^*M \otimes_s T^*M)$. Since trivially $\psi_*^0 g - g = 0$, we may find $\varepsilon > 0$ such that $\psi_*^s g - g$ falls in $GHP(\mathcal{M}, K)$ for each $|s| < \varepsilon$. Defining $h^s = \psi_*^{\varepsilon s} g - g$, we obtain a 1-parameter family $(-1, 1) \rightarrow GHP(\mathcal{M}, K)$, $s \mapsto h^s$ such that $h^0 = 0$. By definition of β , we have

$$\int_M \left(\frac{dh^s}{ds} \Big|_0 \right) (\beta) d\mu_g = \frac{d}{ds} \langle \Omega, \pi(R_{h^s}^{\mathcal{M}} b) \Omega \rangle \Big|_0. \quad (3.1.4)$$

On the one hand $h^s = \psi_*^{\varepsilon s} g - g = \psi_*^{\varepsilon s} g_{h^0} - g$ is an element of $GHP(\mathcal{M}, K)$ for each $s \in (-1, 1)$, hence we can apply Proposition 3.1.4 (we choose $h = h^0 = 0$ as original perturbation and $h' = h^s$ for each $s \in (-1, 1)$) and we deduce that $R_{h^s}^{\mathcal{M}} = R_{h^0}^{\mathcal{M}} = \text{id}_{\mathcal{A}(\mathcal{M})}$ (the last equality follows from the fact that $h^0 = 0$) for each $s \in (-1, 1)$. This fact entails

$$\frac{d}{ds} \langle \Omega, \pi(R_{h^s}^{\mathcal{M}} b) \Omega \rangle \Big|_0 = 0.$$

On the other hand for each $p \in M$ we have

$$\frac{d}{ds} h_{ij}^s(p) \Big|_0 = \varepsilon \frac{d}{ds} (\psi_*^s g)_{ij}(p) \Big|_0 = \varepsilon (\nabla_i X_j + \nabla_j X_i).$$

In fact $\left. \frac{d}{ds} (\psi_*^s g)_{ij} (p) \right|_0$ is exactly the definition of the Lie derivative of g along the vector field X (cfr. [Wal84, eq. C.2.1, p. 439]) and the last equivalence follows from [Wal84, eq. C.2.16, p. 441]. Inserting the last two equations into eq. (3.1.4), we get the following result:

$$\varepsilon \int_V \beta^{ij} (\nabla_i X_j + \nabla_j X_i) \sqrt{|\det g|} dV = 0,$$

where we used the fixed coordinate neighborhood (U, V, ϕ) to express the integral in local coordinates (this can actually be done since the integrand is supported in $K \subseteq U$). With the exception of ε , which can be thrown away being a positive number, this is exactly our last reformulation of the thesis, eq. (3.1.3). \square

3.2 Relative Cauchy evolution for concrete fields

The functional derivative of the relative Cauchy evolution with respect to the space-time metric was defined as a section in $T^*M \otimes_s T^*M$, hence, as we already observed, it is symmetric by construction. Moreover in Proposition 3.1.8 we proved that its divergence is null. Both these properties are good hints to support the conjecture that the functional derivative of the RCE has the meaning of a quantized stress-energy tensor. In this section we settle this question once and for all for the cases of the Klein-Gordon field (already discussed in [BFV03]), the Proca field and the electromagnetic field on a globally hyperbolic spacetime \mathcal{M} .

3.2.1 Quasi-free Hadamard states

This subsection is devoted to introduce quasi-free Hadamard states, an essential ingredient in our way to the proof of the theorems stating the compatibility between the action of the quantized stress-energy tensor of the Klein-Gordon, Proca or electromagnetic field and the functional derivative of the corresponding relative Cauchy evolution with respect to the spacetime metric.

To start, we consider the locally covariant quantum field theory $\mathcal{A} : \mathbf{ghs}^f \rightarrow \mathbf{alg}$ built in Section 2.2 (in the next subsections \mathcal{A} will be one of the LCQFTs built for the concrete examples discussed in Section 2.3) and we choose an object (\mathcal{M}, E, A) of \mathbf{ghs}^f so that we have at our disposal the unital C*-algebra $\mathcal{A}(\mathcal{M})$. Now we take a state τ (see Definition 1.4.17) and we apply Theorem 1.4.22. In this way we obtain the GNS triple $(\mathcal{H}_\tau^\mathcal{M}, \pi_\tau^\mathcal{M}, \Omega_\tau^\mathcal{M})$ associated to the state τ on the unital C*-algebra $\mathcal{A}(\mathcal{M}, E, A)$.

Recalling the procedure of Section 2.2, we see that $\mathcal{A}(\mathcal{M}, E, A)$ is the (unique up to *-isomorphisms) CCR representation (\mathcal{V}, V) of the symplectic space $(V, \sigma) = \mathcal{B}(\mathcal{M}, E, A)$ provided by the covariant functor $\mathcal{B} : \mathbf{ghs}^f \rightarrow \mathbf{ssp}$ describing the

theory of the field at a classical level (as a matter of fact \mathcal{A} was obtained as the composition of \mathcal{B} with the covariant functor $\mathcal{C} : \mathbf{ssp} \rightarrow \mathbf{alg}$ embodying the quantization procedure). We define the represented counterpart of the Weyl map V setting $V_\tau^\mathcal{M} = \pi_\tau^\mathcal{M} \circ V : V \rightarrow \mathcal{B}(\mathcal{H}_\tau^\mathcal{M})$, where $\mathcal{B}(\mathcal{H}_\tau^\mathcal{M})$ denotes the unital C^* -algebra of the linear and continuous operators on the Hilbert space $\mathcal{H}_\tau^\mathcal{M}$. Note that $V_\tau^\mathcal{M}$ maps each $u \in V$ to a unitary operator $V_\tau^\mathcal{M}(u)$ on the Hilbert space $\mathcal{H}_\tau^\mathcal{M}$, as one easily deduces from Remark 1.4.12 and the fact that $\pi_\tau^\mathcal{M}$ is a unit preserving $*$ -homomorphism from \mathcal{V} to $\mathcal{B}(\mathcal{H}_\tau^\mathcal{M})$. With reference to [AG93, Chap VI, Sect. 62, p. 16] and [AG93, Chap VI, Sect. 74, p. 74], we can find a selfadjoint operator $\Phi_\tau^\mathcal{M}(u) \in \mathcal{B}(\mathcal{H}_\tau^\mathcal{M})$ such that

$$e^{i\Phi_\tau^\mathcal{M}(u)} = V_\tau^\mathcal{M}(u). \quad (3.2.1)$$

We may interpret the selfadjoint operator $\Phi_\tau^\mathcal{M}(u)$ as the quantum field corresponding to the classical field $u \in V$. It turns out that a map from V to $\mathcal{B}(\mathcal{H}_\tau^\mathcal{M})$ is automatically defined:

$$\begin{aligned} \Phi_\tau^\mathcal{M} : V &\rightarrow \mathcal{B}(\mathcal{H}_\tau^\mathcal{M}) \\ u &\mapsto \Phi_\tau^\mathcal{M}(u). \end{aligned}$$

Using the map $\Phi_\tau^\mathcal{M}$ we can define the n -point functions on the state τ and then characterize quasi-free states.

Definition 3.2.1. Denote with $\mathcal{A} : \mathbf{ghs}^f \rightarrow \mathbf{alg}$ the locally covariant quantum field theory and with $\mathcal{B} : \mathbf{ghs}^f \rightarrow \mathbf{ssp}$ the covariant functor describing the classical field theory (cfr. Section 2.2). For each object (\mathcal{M}, E, A) of \mathbf{ghs}^f we consider the CCR representation $(\mathcal{V}, V) = \mathcal{A}(\mathcal{M}, E, A)$ and the symplectic space $(V, \sigma) = \mathcal{B}(\mathcal{M}, E, A)$ and for each state τ on the unital C^* -algebra \mathcal{V} we take the (unique up to unitary transformations) GNS triple $(\mathcal{H}_\tau^\mathcal{M}, \pi_\tau^\mathcal{M}, \Omega_\tau^\mathcal{M})$ provided by Theorem 1.4.22 applied to the state τ on the unital C^* -algebra \mathcal{V} . Following the procedure shown above, we obtain a map

$$\begin{aligned} \Phi_\tau^\mathcal{M} : V &\rightarrow \mathcal{B}(\mathcal{H}_\tau^\mathcal{M}) \\ u &\mapsto \Phi_\tau^\mathcal{M}(u) \end{aligned}$$

for each $(\mathcal{M}, E, A) \in \mathbf{Obj}_{\mathbf{ghs}^f}$ and each state τ on the unital C^* -algebra $\mathcal{A}(\mathcal{M}, E, A)$.

We define the n -point function on $\theta \in \mathcal{H}_\tau^\mathcal{M}$ as the map

$$\begin{aligned} w_{\tau,n}^{\mathcal{M},\theta} : \overbrace{\mathcal{D}(M, E) \times \cdots \times \mathcal{D}(M, E)}^{n \text{ times}} &\rightarrow \mathbb{R} \\ (f_1, \dots, f_n) &\mapsto \langle \theta, \Phi_\tau^\mathcal{M}(e_A f_1) \cdots \Phi_\tau^\mathcal{M}(e_A f_n) \theta \rangle_\tau^\mathcal{M}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_\tau^\mathcal{M}$ denotes the scalar product of the Hilbert space $\mathcal{H}_\tau^\mathcal{M}$. The n -point function $w_{\tau,n}^{\mathcal{M},\Omega_\tau^\mathcal{M}}$ on the vector $\Omega_\tau^\mathcal{M}$ of the GNS triple is simply denoted by $w_{\tau,n}^\mathcal{M}$.

We say that the state τ is *quasi-free* if its $(2n + 1)$ -point function $w_{\tau, 2n+1}^{\mathcal{M}}$ vanishes for each $n \in \mathbb{N}$, while its $2n$ -point function satisfies the following identity for each $n \in \mathbb{N}$:

$$w_{\tau, 2n}^{\mathcal{M}}(f_1, \dots, f_{2n}) = \sum_s w_{\tau, 2}^{\mathcal{M}}(f_{s(1)}, f_{s(2)}) \cdots w_{\tau, 2}^{\mathcal{M}}(f_{s(2n-1)}, f_{s(2n)})$$

for each $f_1, \dots, f_{2n} \in \mathcal{D}(M, E)$, where the sum is taken over all the permutations s of $\{1, \dots, 2n\}$ such that $s(1) < s(3) < \dots < s(2n-1)$ and $s(2) < s(4) < \dots < s(2n)$.

Note that for each quasi-free state all the n -point functions $w_{\tau, n}^{\mathcal{M}}$ are completely determined by the 2-point function $w_{\tau, 2}^{\mathcal{M}}$.

Now we want to spend few words about *Hadamard states*. These states are widely accepted as the physically meaningful states for quantum field theories on curved spacetimes. This is due to the fact that the short distance behavior of their 2-point functions mimics the short distance behavior of vacuum states for quantum field theories on Minkowski spacetime. Although singularities are present, they are controlled in such a way that the expectation values of physical observables (e.g. the stress-energy tensor) on Hadamard states are prevented from taking unbounded fluctuations.

To give an idea of what it is meant for a Hadamard state we give the following definition according Kay and Wald, [Kay91]. Indeed this is specific for the Klein-Gordon field, yet it already gives a sketch of the constraints on the singularities of a Hadamard state. A precise extension of the notion of Hadamard state to fields in arbitrary vector bundles can be found in [SV01, Sect. 5.1, p. 20].

Definition 3.2.2. Let $\mathcal{A} : \mathfrak{ghs}^{KG} \rightarrow \mathfrak{alg}$ be the LCQFT for the Klein-Gordon field (cfr. Subsection 2.3.1) and let $(\mathcal{M}, \Lambda^0 M, A)$ be an object of the category \mathfrak{ghs}^{KG} . Consider a diffeomorphism $\psi : M \rightarrow \mathbb{R} \times S$ provided by Theorem 1.2.15 (S is some smooth spacelike Cauchy surface for \mathcal{M}) and define the smooth function $T = \text{pr}_1 \circ \psi : M \rightarrow \mathbb{R}$, where pr_1 denotes the projection on the first argument of the Cartesian product. We define the squared geodesic distance d on an open neighborhood O in $M \times M$ of the set of causally related points (p, q) such that $J_+^{\mathcal{M}}(p) \cap J_-^{\mathcal{M}}(q)$ and $J_-^{\mathcal{M}}(p) \cap J_+^{\mathcal{M}}(q)$ are included in a convex normal neighborhood. It turns out that d is well defined and smooth. Then for each $n \in \{0, 1, 2, \dots\}$ and each $\varepsilon > 0$ we define the function $G_{n, \varepsilon} : O \rightarrow \mathbb{C}$ according to the formula

$$G_{n, \varepsilon}(p, q) = \frac{1}{(2\pi)^2} \left(\frac{\Delta^{1/2}}{\gamma(p, q)} + v^{(n)}(p, q) \ln \gamma(p, q) \right),$$

where the branch-cut for the logarithm is taken on the negative half of the real line,

Δ is the van Vleck-Morette determinant (refer to [DB60]),

$$v^{(n)}(p, q) = \sum_{m=1}^n v_n(p, q) (\sigma(p, q))^m,$$

the functions v_n are uniquely determined via the Hadamard recursion relations (refer to [DB60, Gar98]) and

$$\gamma(p, q) = d(p, q) + 2i\varepsilon(T(p) - T(q)) + \varepsilon^2.$$

Now let Σ be a smooth spacelike Cauchy surface for M and take a causal normal neighborhood of Σ in M (its existence is proved in [Kay91, Lem. 2.2, p. 62]). Consider an open neighborhood O' in $N \times N$ of the set of pairs of causally related points such that the closure of O' in $N \times N$ is contained in O . Let χ be a smooth real valued function on $N \times N$ which is null outside O and equal to 1 inside O' . Then we say that a state τ on the unital C^* -algebra $\mathcal{A}(\mathcal{M}, E, A)$ is a Hadamard state if its 2-point function $w_{\tau,2}^{\mathcal{M}}$ is such that for each $n \in \{0, 1, 2, \dots\}$ there exists a function $H_n \in C^n(N \times N)$ which satisfies the following condition:

$$w_{\tau,2}^{\mathcal{M}}(f_1, f_2) = \lim_{\varepsilon \rightarrow 0} \iint_N \Lambda_{n,\varepsilon}(p, q) f_1(p) f_2(q) d\mu_g(p) d\mu_g(q)$$

for each $f_1, f_2 \in \mathcal{D}(N)$, where

$$\Lambda_{n,\varepsilon}(p, q) = \chi(p, q) G_{n,\varepsilon}(p, q) + H_n(p, q).$$

The problem of the determination of Hadamard states on curved spacetimes for the various quantum fields one may consider is not discussed here, neither we analyze the properties of Hadamard states in detail because this would require the introduction of several notions from microlocal analysis. Anyway we provide some references:

- [Hör90] for the necessary tools of microlocal analysis;
- [Rad96, SV01, SVW02, FV03, San10a] are only some of the publications discussing conditions (in the context of microlocal analysis) for a state on some C^* -algebra that are equivalent to the requirement of being Hadamard (both for the case of a specific fields or for more general situations) and showing the existence of Hadamard states for specific fields.

In the present context we are mainly interested in the existence of Hadamard states for spin 1 fields. Such result was established by Fewster and Pfenning in [FP03].

Anyway few remarks about some of the properties of the GNS representation induced by a Hadamard state are required. Let $\mathcal{A} : \mathfrak{ghs}^f \rightarrow \mathfrak{alg}$ be the LCQFT

built in Section 2.2 (remember that it is causal and, above all, it fulfils the time slice axiom) and let (\mathcal{M}, E, A) be an object of \mathbf{ghs}^f . Consider a Hadamard state τ on the CCR representation $(\mathcal{V}, V) = \mathcal{A}(\mathcal{M}, E, A)$ (recall that $(V, \sigma) = \mathcal{B}(\mathcal{M}, E, A)$ denotes the symplectic space from which (\mathcal{V}, V) arises via the quantization functor \mathcal{C}) and denote with $(\mathcal{H}_\tau^\mathcal{M}, \pi_\tau^\mathcal{M}, \Omega_\tau^\mathcal{M})$ its associated GNS triple. Then the state τ is sufficiently regular to allow us to regard the function

$$t \in \mathbb{R} \mapsto V_\tau^\mathcal{M}(tu)$$

as a differentiable function whatever choice of $u \in V$ we make. This gives us the opportunity to define the map:

$$\begin{aligned} \Psi_\tau^\mathcal{M} : \mathcal{D}(M, E) &\rightarrow \mathcal{B}(\mathcal{H}_\tau^\mathcal{M}) \\ f &\mapsto -i \frac{d}{dt} V_\tau^\mathcal{M}(te_A f) \Big|_0, \end{aligned}$$

where e_A denotes the causal propagator for A . For each $f \in \mathcal{D}(M, E)$ we call $\Psi_\tau^\mathcal{M}(f)$ *smeared field*. We deduce from its definition that the map $\Psi_\tau^\mathcal{M}$ is linear and that the corresponding smeared fields allow us to write $V_\tau^\mathcal{M}(e_A f)$ in exponential form (cfr. eq. (3.2.1)) for each $f \in \mathcal{D}(M, E)$: one easily checks that

$$i\Psi_\tau^\mathcal{M}(f) = \frac{d}{dt} V_\tau^\mathcal{M}(te_A f) \Big|_0$$

agrees with

$$e^{i\Psi_\tau^\mathcal{M}(f)} = V_\tau^\mathcal{M}(e_A f).$$

In this way we can also see that $\Psi_\tau^\mathcal{M}(f) = \Phi_\tau^\mathcal{M}(e_A f)$ for each $f \in \mathcal{D}(M, E)$. Moreover we can deduce the commutation relation between smeared fields from the Weyl relations (cfr. Definition 1.4.11) and also the commutation relation between a smeared field and a represented Weyl generator. We find

$$\begin{aligned} [\Psi_\tau^\mathcal{M}(f), \Psi_\tau^\mathcal{M}(f')] &= i\sigma(e_A f, e_A f'), \\ [\Psi_\tau^\mathcal{M}(f), V_\tau^\mathcal{M}(e_A f')] &= -\sigma(e_A f, e_A f') V_\tau^\mathcal{M}(e_A f') \end{aligned}$$

for each $f, f' \in \mathcal{D}(M, E)$. These relations will be useful in the proof of the theorems stating the agreement between the action of the functional derivative of the relative Cauchy evolution and the quantized stress-energy tensor. In particular it is interesting for this purpose to consider the commutator of the product of two smeared fields with some represented Weyl generator. Exploiting the second commutation

relation given above, we find

$$\begin{aligned} [\Psi_\tau^\mathcal{M}(f) \Psi_\tau^\mathcal{M}(f'), V_\tau^\mathcal{M}(u)] &= -\sigma(e_A f, u) V_\tau^\mathcal{M}(u) \Psi_\tau^\mathcal{M}(f') \\ &\quad - \sigma(e_A f', u) \Psi_\tau^\mathcal{M}(f) V_\tau^\mathcal{M}(u) \end{aligned} \quad (3.2.2)$$

for each $f, f' \in \mathcal{D}(M, E)$ and each $u \in V$.

There is still another very important consequence of the choice of a quasi-free Hadamard state τ : We find a dense subspace $\mathcal{V}_\tau^\mathcal{M}$ of the Hilbert space $\mathcal{H}_\tau^\mathcal{M}$, namely the one constituted by all the vectors obtained applying an arbitrary polynomial in $\Psi_\tau^\mathcal{M}(f)$ and $V_\tau^\mathcal{M}(u)$ (for any choice of $f \in \mathcal{D}(M, E)$ and $u \in V$) to the GNS vector $\Omega_\tau^\mathcal{M}$, and a dense unital sub- $*$ -algebra $\mathcal{B}_\tau^\mathcal{M}$ of $\mathcal{A}(\mathcal{M}, E, A)$ such that Assumption 3.1.5 holds. This fact entails that we can actually give sense to the functional derivative of the RCE.

Moreover it is possible to establish a relation that will be the key for the proof of our theorems from now on. First of all we have to define a classical counterpart of the relative Cauchy evolution which is obtained simply replacing the covariant functor \mathcal{A} with the covariant functor \mathcal{B} in eq. (3.1.1): for each object (\mathcal{M}, E, A) of \mathbf{ghs}^f and each $h \in GHP(\mathcal{M})$ we set

$$r_h^\mathcal{M} = \mathcal{B}(i_-^\mathcal{M}[h]) \circ \mathcal{B}(j_-^\mathcal{M}[h])^{-1} \circ \mathcal{B}(j_+^\mathcal{M}[h]) \circ \mathcal{B}(i_+^\mathcal{M}[h])^{-1}.$$

Note that the definition is well posed because a proper version of the time slice axiom holds also for the covariant functor $\mathcal{B} : \mathbf{ghs}^f \rightarrow \mathbf{ssp}$ describing the classical field theory (see Subsection 2.2.1) and that $R_h^\mathcal{M} = \mathcal{C}(r_h^\mathcal{M})$, where $\mathcal{C} : \mathbf{ssp} \rightarrow \mathbf{alg}$ is the covariant functor that realizes the quantization procedure (cfr. Subsection 2.2.2).² With this definition we are ready to state the key relation which can be found in [FP03, Prop. A.8, p. 363]:

$$\frac{d}{ds} \left\langle \theta, V_\tau^\mathcal{M}(r_{h^s}^\mathcal{M} u) \theta \right\rangle_\tau^\mathcal{M} \Big|_0 = \frac{i}{2} \left\langle \theta, \left\{ \Phi_\tau^\mathcal{M} \left(\frac{d}{ds} (r_{h^s}^\mathcal{M} u) \Big|_0 \right), V_\tau^\mathcal{M}(u) \right\} \theta \right\rangle_\tau^\mathcal{M} \quad (3.2.3)$$

for each compact subset K of M , each smooth 1-parameter family of globally hyperbolic perturbations $(-1, 1) \rightarrow GHP(\mathcal{M}, K)$, $s \mapsto h^s$ such that $h^0 = 0$, each $\theta \in \mathcal{V}_\tau^\mathcal{M}$ and each $u \in V$. In [FV03] the proof is performed in the context of the Klein-Gordon field, however it holds in general since it relies only on the properties of the CCR representation of some symplectic space and on the choice of a Hadamard state which gives rise to a Hilbert space representation with the “good” properties mentioned above.

In the upcoming subsections, in which we deal with concrete fields, we will always fix a quasi-free Hadamard state on the unital C^* -algebra provided by the LCQFT

²In the following we will study in some detail the classical RCE for the specific fields we will consider.

for such field on some globally hyperbolic spacetime (note that all the LCQFTs we built in Section 2.3 fulfil the time slice axiom) and we will perform calculations exploiting all the properties that we presented here.

3.2.2 Relative Cauchy evolution for the Klein-Gordon field

In this subsection we follow the calculations in [BFV03] to show a relation between the functional derivative of the relative Cauchy evolution for the Klein-Gordon field and its quantized stress-energy tensor. This relation will be proved in the theorem concluding this subsection. First of all we need to introduce all the building blocks.

Relative Cauchy evolution for the classical Klein-Gordon field

As a starting point we consider Subsection 2.3.1, where we discussed the construction of a locally covariant quantum field theory for the Klein-Gordon field applying a specialization of the general procedure (Section 2.2). Here we use the notation introduced in Subsection 2.3.1 and in Section 2.2.

The first ingredient that we need to consider pertains to the classical theory of the Klein-Gordon field. Denote with $\mathcal{B} : \mathbf{ghs}^{KG} \rightarrow \mathbf{ssp}$ the covariant functor describing the classical theory of the Klein-Gordon field built following the procedure of Subsection 2.2.1 (specialized to the case of the Klein-Gordon field along the line sketched in Subsection 2.3.1). In the upcoming proposition it appears an almost self-explanatory notation, namely we write $A|_O$, where A is the formally selfadjoint normally hyperbolic operator $\square_0 + m^2 \text{id}_{\Omega^0 M}$ governing the Klein-Gordon field. In any case the beginning of the proof clarifies precisely what $A|_O$ stands for.

Proposition 3.2.3. *Let $\mathcal{B} : \mathbf{ghs}^{KG} \rightarrow \mathbf{ssp}$ be the covariant functor describing the classical theory of the Klein-Gordon field, let $(\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t}), \Lambda^0 M, A)$ be an object of \mathbf{ghs}^{KG} and let O be an \mathcal{M} -causally convex connected open subset of M including a smooth spacelike Cauchy surface Σ for \mathcal{M} . Consider $(\mathcal{M}|_O, \Lambda^0 O, A|_O) \in \mathbf{Obj}_{\mathbf{ghs}^{KG}}$ and the morphism $(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M})$ of \mathbf{ghs}^{KG} from $(\mathcal{M}|_O, \Lambda^0 O, A|_O)$ to $(\mathcal{M}, \Lambda^0 M, A)$ induced by the inclusion maps $\iota_O^M : O \rightarrow M$ and $\iota_{\Lambda^0 O}^{\Lambda^0 M} : \Lambda^0 O \rightarrow \Lambda^0 M$. Then there exists a partition of unity $\{\chi^a, \chi^r\}$ on M such that the inverse $\mathcal{B}(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M})^{-1}$ of the bijective morphism $\mathcal{B}(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M})$ of \mathbf{ssp} from $(V, \sigma) = \mathcal{B}(\mathcal{M}|_O, \Lambda^0 O, A|_O)$ to $(W, \omega) = \mathcal{B}(\mathcal{M}, \Lambda^0 M, A)$ satisfies the following equation:*

$$\mathcal{B}(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M})^{-1} \varphi = \pm e_{A|_O} \left(\text{res}_{\iota_{\Lambda^0 O}^{\Lambda^0 M}} (A(\chi^{a/r} \varphi)) \right) \quad \forall \varphi \in W,$$

where $e_{A|_O}$ is the causal propagator for $A|_O$ and the restriction map is defined in Lemma 2.2.4.

Proof. We fix a globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ and we consider an \mathcal{M} -causally convex connected open subset O of M including a smooth spacelike

Cauchy surface Σ for \mathcal{M} . In Remark 2.1.2 we saw that we can consider the globally hyperbolic spacetime $\mathcal{M}|_O$ and that the inclusion map $\iota_O^M : O \rightarrow M$ can be interpreted as a morphism of \mathbf{ghs} from $\mathcal{M}|_O$ to \mathcal{M} . Exploiting 1.1.14, we realize that $\Lambda^0 M|_O = \Lambda^0 O$ is a vector bundle and that $(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M})$ is a vector bundle homomorphism. It follows from the comments made after Definition 2.2.1 that $\Lambda^0 O$ can be endowed with the restriction of the inner product on $\Lambda^0 M$ and that we can consider the formally selfadjoint normally hyperbolic operator $A_{\iota_{\Lambda^0 O}^{\Lambda^0 M}|_O}$ (for convenience we denote it with $A|_O$). Hence we have the object $(\mathcal{M}|_O, \Lambda^0 O, A|_O)$ of \mathbf{ghs}^{KG} and, exploiting again the comments made after Definition 2.2.1, we immediately see that $(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M})$ is a morphism of \mathbf{ghs}^{KG} from $(\mathcal{M}|_O, \Lambda^0 O, A|_O)$ to $(\mathcal{M}, \Lambda^0 M, A)$.

Now the main part of the proof begins. We exploit [BS06, Thm. 1.2] that provides us (among other things) a diffeomorphism $\psi : M \rightarrow \mathbb{R} \times \Sigma$ such that $\psi^{-1}(\{0\} \times \Sigma) = \Sigma$ and $\Sigma_t = \psi(\{t\} \times \Sigma)$ is a smooth spacelike Cauchy surface for \mathcal{M} for each $t \in \mathbb{R}$. Since Σ is included in O by hypothesis and O is open, we deduce that O is a neighborhood of Σ . ψ^{-1} is continuous, therefore we find $\varepsilon > 0$ such that $\psi^{-1}([-\varepsilon, \varepsilon] \times \Sigma) \subseteq O$. This entails that $\Sigma_{-\varepsilon}$ and Σ_ε are smooth spacelike Cauchy surfaces for \mathcal{M} that are contained in O . We consider the open covering $\{I_+^{\mathcal{M}}(\Sigma_{-\varepsilon}), I_-^{\mathcal{M}}(\Sigma_\varepsilon)\}$ of M and its subordinate partition of unity $\{\chi^a, \chi^r\}$.

Take $\varphi \in W$ and denote the causal propagator for the formally selfadjoint normally hyperbolic operator $A = \square_0 + m^2 \text{id}_{\Omega^0 M}$ with e_A . As a consequence of the construction of the functor \mathcal{B} , $W = e_A(\Omega_0^0 M)$. Hence there exists a compact subset K of M such that $\text{supp}(\varphi) \subseteq J^{\mathcal{M}}(K)$. We define $\varphi^{a/r} = \chi^{a/r} \varphi$:

$$\text{supp}(\varphi^{a/r}) \subseteq J_{\pm}^{\mathcal{M}}(\Sigma_{\mp \varepsilon}).$$

We deduce that $\varphi^{a/r}$ is an element of $\Omega^0 M$ with \mathcal{M} -past/future compact support. Another consequence of $W = e_A(\Omega_0^0 M)$ is $A\varphi = 0$. From this fact, together with $\chi^a + \chi^r = 1$, we deduce $A\varphi^a = -A\varphi^r$, hence

$$\text{supp}(A\varphi^a) \subseteq J^{\mathcal{M}}(K) \cap J_+^{\mathcal{M}}(\Sigma_{-\varepsilon}) \cap J_-^{\mathcal{M}}(\Sigma_\varepsilon) \subseteq O.$$

Exploiting Proposition 1.2.18, we realize that $A\varphi^{a/r} \in \Omega_0^0 M$ with support contained in O . At this point we can apply the restriction map (its definition in the general context of arbitrary vector bundles can be found in Lemma 2.2.4) in order to obtain

$$\text{res}_{\iota_{\Lambda^0 O}^{\Lambda^0 M}}(A(\chi^{a/r} \varphi)) \in \Omega_0^0 O.$$

Therefore it makes sense to consider

$$\pm e_{A|_O} \left(\text{res}_{\iota_{\Lambda^0 O}^{\Lambda^0 M}}(A(\chi^{a/r} \varphi)) \right).$$

This shows that the map

$$\begin{aligned}\alpha : W &\rightarrow V \\ \varphi &\mapsto \pm e_{A|_O} \left(\text{res}_{\iota_{\Lambda^0 O}^{\Lambda^0 M}} (A(\chi^{a/r} \varphi)) \right)\end{aligned}$$

is well defined.

Note that, from the hypothesis made, we know that the image $\iota_O^M(O) = O$ includes a smooth spacelike Cauchy surface for \mathcal{M} . Hence $\mathcal{B} \left(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M} \right)^{-1}$ is a morphism of \mathbf{ssp} from (W, ω) to (V, σ) because the time slice axiom holds for \mathcal{B} (cfr. Theorem 2.2.6). We must check that $\alpha = \mathcal{B} \left(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M} \right)^{-1}$. Take $\varphi \in W$, recall Lemma 2.2.5 and observe that the restriction followed by the corresponding extension leaves the argument of the restriction unchanged:

$$\begin{aligned}\mathcal{B} \left(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M} \right) (\alpha \varphi) &= \mathcal{B} \left(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M} \right) \left(\pm e_{A|_O} \left(\text{res}_{\iota_{\Lambda^0 O}^{\Lambda^0 M}} (A\varphi^{a/r}) \right) \right) \\ &= \pm e_A \left(\text{ext}_{\iota_{\Lambda^0 O}^{\Lambda^0 M}} \left(\text{res}_{\iota_{\Lambda^0 O}^{\Lambda^0 M}} (A\varphi^{a/r}) \right) \right) \\ &= \pm e_A A\varphi^{a/r}.\end{aligned}$$

The support properties of $\varphi^{a/r}$ and $A\varphi^{a/r}$ allow us to apply Lemma 1.3.17:

$$\pm e_A A\varphi^{a/r} = \pm (e_A^a A\varphi^{a/r} - e_A^r A\varphi^{a/r}) = \varphi^a + \varphi^r = \varphi.$$

With this we conclude that

$$\mathcal{B} \left(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M} \right) (\alpha \varphi) = \varphi = \mathcal{B} \left(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M} \right) \left(\mathcal{B} \left(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M} \right)^{-1} \varphi \right) \quad \forall \varphi \in W.$$

Since $\mathcal{B} \left(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M} \right)$ is injective, the last equation entails

$$\alpha \varphi = \mathcal{B} \left(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M} \right)^{-1} \varphi \quad \forall \varphi \in W,$$

therefore we realize that the thesis actually holds. \square

Now we specialize the definition of the RCE to the case of the Klein-Gordon field. Consider an object $(\mathcal{M}, \Lambda^0 M, A)$ of \mathbf{ghs}^{KG} , take $h \in GHP(\mathcal{M})$ and recall the definitions of the morphisms $\iota_{\pm}^{\mathcal{M}}[h]$ and $j_{\pm}^{\mathcal{M}}[h]$ introduced before Definition 3.1.3. Together with the perturbed spacetime $\mathcal{M}[h]$, we must also consider the effect of the perturbation of the spacetime metric on the vector bundle (especially the inner product defined on it) and on the differential operator $A = \square_0 + m^2 \text{id}_{\Omega^0 M}$. In this case $\Lambda^0 M$ and its inner product (being simply the fiberwise multiplication of real numbers) remain unchanged, while we define $A[h] = \square_0[h] + m^2 \text{id}_{\Omega^0 M}$, where $\square_0[h]$ is the d'Alembert operator defined on $\mathcal{M}[h]$ for 0-forms, specifically the metric involved here is $g_h = g + h$ in place of g . We may consider the inclusion map $\iota_{\Lambda^0 M_{\pm}}^{\Lambda^0 M}$,

where $M_{\pm} = M \setminus J_{\mp}^{\mathcal{M}}(\text{supp}(h))$ in accordance with the definitions of $i_{\pm}^{\mathcal{M}}[h]$ and $j_{\pm}^{\mathcal{M}}[h]$. $A|_{M_{\pm}}$ is compatible with A via $(\iota_{M_{\pm}}^M, \iota_{\Lambda^0 M_{\pm}}^{\Lambda^0 M})$ (see the comments made after Definition 2.2.1):

$$\text{ext}_{\iota_{\Lambda^0 M_{\pm}}^{\Lambda^0 M}} \left(A|_{M_{\pm}} f \right) = A \left(\text{ext}_{\iota_{\Lambda^0 M_{\pm}}^{\Lambda^0 M}} f \right) \quad \forall f \in \Omega_0^0 M_{\pm}.$$

Since the effects of the perturbation h are relevant only inside $\text{supp}(h)$, we realize that $A[h]$ and A act exactly in the same way on sections supported outside $\text{supp}(h)$. Together with $A|_{M_{\pm}}$, we may consider $A[h]|_{M_{\pm}}$ and we immediately recognize that they coincide (we denote both of them with $A_{\pm}[h]$ in a fashion similar to that used when we introduced $\mathcal{M}_{\pm}[h]$ to denote $\mathcal{M}|_{M_{\pm}} = \mathcal{M}[h]|_{M_{\pm}}$). All these observations are made in order to introduce the objects $(\mathcal{M}[h], \Lambda^0 M, A[h])$ and $(\mathcal{M}_{\pm}[h], \Lambda^0 M_{\pm}, A_{\pm}[h])$ of \mathfrak{ghs}^{KG} and to interpret the vector bundle homomorphism $(\iota_{M_{\pm}}^M, \iota_{\Lambda^0 M_{\pm}}^{\Lambda^0 M}) : \Lambda^0 M_{\pm} \rightarrow \Lambda^0 M$ in the following (generally inequivalent) ways (note the analogy with the definitions of $i_{\pm}^{\mathcal{M}}[h]$ and $j_{\pm}^{\mathcal{M}}[h]$ as different morphisms obtained from the same inclusion map $\iota_{M_{\pm}}^M$):

$$\begin{aligned} \left(i_{\pm}^{\mathcal{M}}[h], i_{\pm}^{\mathcal{M}, \Lambda^0}[h] \right) &\in \text{Mor}_{\mathfrak{ghs}^{KG}} \left((\mathcal{M}_{\pm}[h], \Lambda^0 M_{\pm}, A_{\pm}[h]), (\mathcal{M}, \Lambda^0 M, A) \right), \\ \left(j_{\pm}^{\mathcal{M}}[h], j_{\pm}^{\mathcal{M}, \Lambda^0}[h] \right) &\in \text{Mor}_{\mathfrak{ghs}^{KG}} \left((\mathcal{M}_{\pm}[h], \Lambda^0 M_{\pm}, A_{\pm}[h]), (\mathcal{M}[h], \Lambda^0 M, A[h]) \right). \end{aligned}$$

Denote with \mathcal{A} the LCQFT (fulfilling both the causality condition and the time slice axiom) built following the procedure of Section 2.2 specialized according to Subsection 2.3.1. For $(\mathcal{M}, \Lambda^0 M, A) \in \text{Obj}_{\mathfrak{ghs}^{KG}}$ and $h \in GHP(\mathcal{M})$ we define the RCE for the Klein-Gordon field as:

$$\begin{aligned} R_h^{\mathcal{M}} &= \mathcal{A} \left(i_{-}^{\mathcal{M}}[h], i_{-}^{\mathcal{M}, \Lambda^0}[h] \right) \circ \mathcal{A} \left(j_{-}^{\mathcal{M}}[h], j_{-}^{\mathcal{M}, \Lambda^0}[h] \right)^{-1} \\ &\quad \circ \mathcal{A} \left(j_{+}^{\mathcal{M}}[h], j_{+}^{\mathcal{M}, \Lambda^0}[h] \right) \circ \mathcal{A} \left(i_{+}^{\mathcal{M}}[h], i_{+}^{\mathcal{M}, \Lambda^0}[h] \right)^{-1}. \end{aligned}$$

In a similar way one can consider a classical version of the RCE based on the covariant functor \mathcal{B} describing the classical theory of the Klein-Gordon field (this is actually possible due to version of the time slice axiom satisfied by \mathcal{B} , cfr. Theorem 2.2.6):

$$\begin{aligned} r_h^{\mathcal{M}} &= \mathcal{B} \left(i_{-}^{\mathcal{M}}[h], i_{-}^{\mathcal{M}, \Lambda^0}[h] \right) \circ \mathcal{B} \left(j_{-}^{\mathcal{M}}[h], j_{-}^{\mathcal{M}, \Lambda^0}[h] \right)^{-1} \\ &\quad \circ \mathcal{B} \left(j_{+}^{\mathcal{M}}[h], j_{+}^{\mathcal{M}, \Lambda^0}[h] \right) \circ \mathcal{B} \left(i_{+}^{\mathcal{M}}[h], i_{+}^{\mathcal{M}, \Lambda^0}[h] \right)^{-1}. \end{aligned}$$

Since the LCQFT \mathcal{A} is obtained via composition of \mathcal{B} with the quantization functor \mathcal{C} presented in Subsection 2.2.2, we almost immediately realize that

$$R_h^{\mathcal{M}} = \mathcal{C} \left(r_h^{\mathcal{M}} \right) \quad (3.2.4)$$

(this is simply a consequence of the covariant axioms fulfilled by any covariant functor). We can determine the action of $r_h^{\mathcal{M}}$ applying Proposition 3.2.3 and Lemma 2.2.5. We find proper partitions of unity $\{\chi_+^a, \chi_+^r\}$ and $\{\chi_-^a, \chi_-^r\}$ on M such that we can express $\mathcal{B}\left(i_+^{\mathcal{M}}[h], i_+^{\mathcal{M}, \Lambda^0}[h]\right)^{-1}$ and respectively $\mathcal{B}\left(j_-^{\mathcal{M}}[h], j_-^{\mathcal{M}, \Lambda^0}[h]\right)^{-1}$ according to Proposition 3.2.3. If we take $\varphi \in \mathcal{B}(\mathcal{M}, \Lambda^0 M, A)$ and evaluate $r_h^{\mathcal{M}}\varphi$, we easily obtain the following result:

$$r_h^{\mathcal{M}}\varphi = e_A A[h] \left(\chi_-^{a/r} e_{A[h]} A \left(\chi_+^{a/r} \varphi \right) \right). \quad (3.2.5)$$

In the following we will need the expression of the derivative $\frac{d}{ds} r_{h^s}^{\mathcal{M}} \varphi|_0$ for an arbitrary smooth 1-parameter family of perturbations of the metric. For convenience in the upcoming calculation we write δ_s in place of $\frac{d}{ds}(\cdot)|_0$. Fix now $\varphi \in \mathcal{B}(\mathcal{M}, \Lambda^0 M, A)$, a compact subset K of M and a smooth 1-parameter family of globally hyperbolic perturbations $(-1, 1) \rightarrow GHP(\mathcal{M}, K)$, $s \mapsto h^s$ and evaluate $\delta_s r_{h^s}^{\mathcal{M}} \varphi$. Our starting point is eq. (3.2.5) with the choice of the superscript r (if we choose a , we face a very similar calculation and we indeed obtain the same result). In the present situation apparently we would have to consider different partitions of unity $\{\chi_+^a, \chi_+^r\}$ and $\{\chi_-^a, \chi_-^r\}$ for each of the values assumed by s . Anyway this difficulty can be avoided making an intelligent choice of the smooth spacelike Cauchy surfaces used to define the partitions of unity: We use always the same foliation of \mathcal{M} (induced by some fixed smooth Cauchy surface Σ for \mathcal{M}) and take the smooth spacelike Cauchy surfaces inside $M_{\pm} = M \setminus J_{\mp}^{\mathcal{M}}(K)$ instead of choosing, for each value of s , a pair of proper smooth spacelike Cauchy surfaces inside $M_{\pm} = M \setminus J_{\mp}^{\mathcal{M}}(\text{supp}(h^s))$. In this way a single choice of the smooth spacelike Cauchy surfaces is satisfactory for each value of s . Such choice is possible because the supports of all the elements h^s in the family of perturbations are controlled by the compact subset K of M .

In the first step we apply the Leibniz rule³:

$$\delta_s r_{h^s}^{\mathcal{M}} \varphi = e_A \left((\delta_s A[h^s]) \left(\chi_-^r e_{A[h^s]} A \left(\chi_+^r \varphi \right) \right) + A \left(\chi_-^r (\delta_s e_{A[h^s]}) A \left(\chi_+^r \varphi \right) \right) \right).$$

We focus on the first addend: On the one hand, following the proof of Proposition 3.2.3 (we are considering $M_- = M \setminus J_+^{\mathcal{M}}(K)$ as O), we can easily see that $\text{supp}(\chi_-^r) \subseteq J_-^{\mathcal{M}}(M_-)$, while on the other hand $\delta_s A[h^s]$ can have coefficients different from zero only inside K . This entails that

$$(\delta_s A[h^s]) \left(\chi_-^r e_{A[h^s]} A \left(\chi_+^r \varphi \right) \right) = 0,$$

³note that causal propagators are sequentially continuous with respect to a proper notion of convergence, refer to [BGP07, Def. 3.4.6, p. 90 and Prop. 3.4.8, p. 91]

therefore we obtain

$$\delta_s r_{h^s}^{\mathcal{M}} \varphi = e_A A \left(\chi_-^r \left(\delta_s e_{A[h^s]} \right) A \left(\chi_+^r \varphi \right) \right).$$

Recalling again the proof of Proposition 3.2.3, we realize that $A \left(\chi_+^r \varphi \right) = -A \left(\chi_+^a \varphi \right)$ and deduces that its support is compact and lies in the causal future of a smooth spacelike Cauchy surface for \mathcal{M} included in $M_+ = M \setminus J_-^{\mathcal{M}}(K)$ (that by construction lies outside K and intersects its causal future). On the contrary χ_-^r is supported in the causal past of a smooth spacelike Cauchy surface for \mathcal{M} included in M_- (that by construction lies outside K and intersects its causal past). These observations entail that $\chi_-^r e_{A[h^s]}^a A \left(\chi_+^r \varphi \right)$ has empty support, hence it is null. Therefore from the last equation we obtain

$$\delta_s r_{h^s}^{\mathcal{M}} \varphi = -e_A A \left(\chi_-^r \left(\delta_s e_{A[h^s]}^r \right) A \left(\chi_+^r \varphi \right) \right). \quad (3.2.6)$$

Now we take a closer look to the term $e_{A[h^s]}^r A[h^s] \left(\chi_+^r \varphi \right)$. In order for this term to make sense it must be shown that $A[h^s] \left(\chi_+^r \varphi \right)$ has compact support. This follows from the the following facts:

- $A \left(\chi_+^r \varphi \right) = -A \left(\chi_+^a \varphi \right)$ implies that $A \left(\chi_+^{a/r} \varphi \right)$ has compact support (note that $\chi_+^{a/r}$ is supported in the causal future/past of a proper smooth spacelike Cauchy surface for \mathcal{M} and remember that $\text{supp}(\varphi) \subseteq J^{\mathcal{M}}(K')$ for a proper compact subset K' of M);
- $A[h^s]$ differs from A only inside K , which is compact.

This two facts imply that

$$\text{supp} \left(A[h^s] \left(\chi_+^{a/r} \varphi \right) \right) \subseteq \text{supp} \left(A \left(\chi_+^{a/r} \varphi \right) \right) \cup K,$$

hence $A[h^s] \left(\chi_+^{a/r} \varphi \right)$ is compactly supported too. From the first point above it follows also that $\chi_+^{a/r} \varphi$ has past/future compact support (we are exploiting Proposition 1.2.18). Hence we can apply Lemma 1.3.17 to conclude that for each s we have

$$e_{A[h^s]}^{a/r} A[h^s] \left(\chi_+^{a/r} \varphi \right) = \chi_+^{a/r} \varphi.$$

Exploiting the Leibniz rule, we find

$$0 = \delta_s \left(\chi_+^r \varphi \right) = \delta_s \left(e_{A[h^s]}^r A[h^s] \left(\chi_+^r \varphi \right) \right) = \left(\delta_s e_{A[h^s]}^r \right) A \left(\chi_+^r \varphi \right) + e_A^r \left(\delta_s A[h^s] \right) \left(\chi_+^r \varphi \right).$$

With this identity we can rewrite eq. (3.2.6):

$$\delta_s r_{h^s}^{\mathcal{M}} \varphi = e_A A \left(\chi_-^r e_A^r \left(\delta_s A[h^s] \right) \left(\chi_+^r \varphi \right) \right).$$

Notice that $(\delta_s A[h^s])(\chi_+^a \varphi) = 0$ because the coefficients of $\delta_s A[h^s]$ are supported inside K while χ_+^a is supported in the causal future of M_+ . Hence we can add such term without modifying the result:

$$(\delta_s A[h^s])(\chi_+^r \varphi) = (\delta_s A[h^s])(\chi_+^r \varphi) + (\delta_s A[h^s])(\chi_+^a \varphi) = (\delta_s A[h^s]) \varphi.$$

In this way we obtain

$$\delta_s r_h^{\mathcal{M}} \varphi = e_A A (\chi_-^r e_A^r (\delta_s A[h^s]) \varphi).$$

Take into account the term $\chi_-^r e_A^a (\delta_s A[h^s]) \varphi$: the coefficients of $\delta_s A[h^s]$ are supported inside K , hence

$$\text{supp}(e_A^a (\delta_s A[h^s]) \varphi) \subseteq J_+^{\mathcal{M}}(K),$$

while χ_-^r is supported inside $J_-^{\mathcal{M}}(M_-)$. This entails that $\chi_-^r e_A^a (\delta_s A[h^s]) \varphi = 0$, therefore we can modify again our last equation with the subtraction of this term leaving the result unchanged:

$$\begin{aligned} \delta_s r_h^{\mathcal{M}} \varphi &= e_A A (\chi_-^r e_A^r (\delta_s A[h^s]) \varphi) - e_A A (\chi_-^r e_A^a (\delta_s A[h^s]) \varphi) \\ &= -e_A A (\chi_-^r e_A (\delta_s A[h^s]) \varphi). \end{aligned}$$

The observation about the support of the coefficients appearing in the linear differential operator $\delta_s A[h^s]$ entails that $f = (\delta_s A[h^s]) \varphi$ is an element of $\Omega_0^0 M$ with support included in K and trivially we have $A e_A f = 0$, so that

$$A (\chi_-^r e_A f) = -A (\chi_-^a e_A f).$$

On account of the last identity, the inclusion $\text{supp}(e_A f) \subseteq J_-^{\mathcal{M}}(K)$ and the identity $\text{supp}(\chi_-^{a/r}) = J_{\pm}^{\mathcal{M}}(\Sigma_-^{a/r})$ for proper smooth spacelike Cauchy surfaces $\Sigma_-^{a/r}$ and applying Proposition 1.2.18 and Lemma 1.3.17, we obtain the following result:

$$-e_A A (\chi_-^r e_A f) = e_A^a A (\chi_-^a e_A f) + e_A^r A (\chi_-^r e_A f) = \chi_-^a e_A f + \chi_-^r e_A f = e_A f.$$

With the last identity we conclude

$$\left. \frac{d}{ds} r_h^{\mathcal{M}} \varphi \right|_0 = e_A \left(\left. \frac{d}{ds} A[h^s] \right|_0 \right) \varphi. \quad (3.2.7)$$

We are left with the problem of the expression for $\delta_s A[h^s] \varphi$. We know that $A[h^s] = \square_0[h^s] + m^2 \text{id}_{\Omega^0 M}$, where $\square_0[h^s]$ denotes the d'Alembert operator built with the perturbed metric g_{h^s} . Indeed the term $m^2 \varphi$ gives null contribution to $\delta_s A[h^s] \varphi$, hence we are interested in the evaluation of $\delta_s \square_0[h^s] \varphi$. Using an arbitrary

coordinate neighborhood, we see from eq. (2.3.2) that

$$\square_0 [h^s] \varphi = -g_{h^s}^{ij} \nabla [h^s]_i \partial_j \varphi = -g_{h^s}^{ij} \partial_i \partial_j \varphi + g_{h^s}^{ij} \Gamma [h^s]_{ij}^k \partial_k \varphi,$$

where $\Gamma [h^s]_{ij}^k$ are the Christoffel symbols of the Levi-Civita connection $\nabla [h^s]$ on $\mathcal{M} [h^s]$, and therefore

$$\left. \frac{d}{ds} A [h^s] \varphi \right|_0 = \left. \frac{d}{ds} \square_0 [h^s] \varphi \right|_0 = \left. \frac{d}{ds} h_{ij}^s \right|_0 \nabla^j \nabla^j \varphi + \left. \frac{d}{ds} \Gamma [h^s]_{ij}^k \right|_0 g^{ij} \nabla_k \varphi, \quad (3.2.8)$$

where in the last step we exploited the relation

$$\left. \frac{d}{ds} h_{ij}^s \right|_0 = \left. \frac{d}{ds} g_{h^s ij} \right|_0 = - \left. \frac{d}{ds} g_{h^s}^{kl} \right|_0 g_{ki} g_{lj} \quad (3.2.9)$$

that follows from $g_{h^s} = g + h^s$ and $g_{h^s}^{kl} g_{h^s ki} g_{h^s lj} = g_{h^s ij}$.

Properties of the GNS representation induced by a quasi-free Hadamard state for the Klein-Gordon field

The second preparatory step is the choice of a quasi-free Hadamard state τ for the unital C*-algebra $(\mathcal{V}, V) = \mathcal{A}(\mathcal{M}, \Lambda^0 M, A)$ (which is actually a CCR representation) describing the Klein-Gordon field on the globally hyperbolic spacetime \mathcal{M} . With this choice, we introduce the (unique up to unitary equivalence) GNS triple $(\mathcal{H}_\tau^\mathcal{M}, \pi_\tau^\mathcal{M}, \Omega_\tau^\mathcal{M})$ induced by τ and we follow the discussion made in Subsection 3.2.1. In this way we obtain the represented version

$$V_\tau^\mathcal{M} = \pi_\tau^\mathcal{M} \circ V : V \rightarrow \mathcal{B}(\mathcal{H}_\tau^\mathcal{M}) \quad (3.2.10)$$

of the Weyl map V , where $(V, \sigma) = \mathcal{B}(\mathcal{M}, \Lambda^0 M, A)$ is the symplectic space provided by the covariant functor \mathcal{B} describing the classical theory of the Klein-Gordon field, together with the map

$$\begin{aligned} \Phi_\tau^\mathcal{M} : V &\rightarrow \mathcal{B}(\mathcal{H}_\tau^\mathcal{M}) \\ \varphi &\mapsto \Phi_\tau^\mathcal{M}(\varphi) \end{aligned}$$

that allows us to express the unitary operator $V_\tau^\mathcal{M}(\varphi)$ as the complex exponential of a selfadjoint operator for each $\varphi \in V$, namely $\Phi_\tau^\mathcal{M}(\varphi)$ is selfadjoint and satisfies $e^{i\Phi_\tau^\mathcal{M}(\varphi)} = V_\tau^\mathcal{M}(\varphi)$. Together with this map, we have the smeared fields (by virtue of the choice of a Hadamard state):

$$\begin{aligned} \Psi_\tau^\mathcal{M} : \Omega_0^0 M &\rightarrow \mathcal{B}(\mathcal{H}_\tau^\mathcal{M}) \\ f &\mapsto -i \left. \frac{d}{dt} V_\tau^\mathcal{M}(te_A f) \right|_0. \end{aligned}$$

As for the general case, it holds that

$$\Psi_{\tau}^{\mathcal{M}}(f) = \Phi_{\tau}^{\mathcal{M}}(e_A f) \quad (3.2.11)$$

for each $f \in \Omega_0^0 M$ and we recognize $\Psi_{\tau}^{\mathcal{M}}$ to be linear.

We stated the most relevant consequence of the choice of a quasi-free Hadamard state τ in Subsection 3.2.1:

- Assumption 3.1.5 is verified, i.e. we find a dense subspace $\mathcal{V}_{\tau}^{\mathcal{M}}$ of $\mathcal{H}_{\tau}^{\mathcal{M}}$ and a dense sub- $*$ -algebra $\mathcal{B}_{\tau}^{\mathcal{M}}$ of $\mathcal{A}(\mathcal{M}, \Lambda^0 M, A)$ such that the functional derivative of the RCE with respect to the spacetime metric can be defined;
- specializing eq. (3.2.3) to the case of the Klein-Gordon field, we see that the following equation holds for each $\theta \in \mathcal{V}_{\tau}^{\mathcal{M}}$, each $\varphi \in V$, each compact subset K of M and each smooth 1-parameter family $(-1, 1) \rightarrow GHP(\mathcal{M}, K)$, $s \mapsto h^s$ such that $h^0 = 0$:

$$\frac{d}{ds} \langle \theta, V_{\tau}^{\mathcal{M}}(r_{h^s}^{\mathcal{M}} \varphi) \theta \rangle_{\tau}^{\mathcal{M}} \Big|_0 = \frac{i}{2} \left\langle \theta, \left\{ \Phi_{\tau}^{\mathcal{M}} \left(\frac{d}{ds} (r_{h^s}^{\mathcal{M}} \varphi) \Big|_0 \right), V_{\tau}^{\mathcal{M}}(\varphi) \right\} \theta \right\rangle_{\tau}^{\mathcal{M}}. \quad (3.2.12)$$

Finally one can show that for each $\eta, \xi \in \mathcal{V}_{\tau}^{\mathcal{M}}$ there exists a smooth section, that we denote with

$$\begin{aligned} M &\rightarrow \mathbb{C} \\ p &\mapsto \langle \eta, \Psi_{\tau}^{\mathcal{M}}(p) \xi \rangle_{\tau}^{\mathcal{M}}, \end{aligned} \quad (3.2.13)$$

where $\langle \cdot, \cdot \rangle_{\tau}^{\mathcal{M}}$ denotes the scalar product of the Hilbert space $\mathcal{H}_{\tau}^{\mathcal{M}}$, such that

$$\langle \eta, \Psi_{\tau}^{\mathcal{M}}(f) \xi \rangle_{\tau}^{\mathcal{M}} = \int_M \langle \eta, \Psi_{\tau}^{\mathcal{M}}(p) \xi \rangle_{\tau}^{\mathcal{M}} f(p) d\mu_g \quad (3.2.14)$$

for each $f \in \Omega_0^0 M$, where $d\mu_g$ is the standard volume form on \mathcal{M} . We may regard this section as (the matrix element of) the unsmeared field. Uniqueness of the unsmeared field is a direct consequence of the last equation.

Quantized stress-energy tensor for the Klein-Gordon field

We still need to find the expression of the quantized stress-energy tensor. This is obtained starting from the action of the Klein-Gordon field on the globally hyperbolic spacetime \mathcal{M} , which in turn comes from the differential operator $A = \square_0 + m^2 \text{id}_{\Omega^0 M}$ governing the field:

$$S_{\mathcal{M}} = \frac{1}{2} (\varphi, A\varphi)_{g,0} = \frac{1}{2} (d\varphi, d\varphi)_{g,1} + \frac{1}{2} m^2 (\varphi, \varphi)_{g,0} = \frac{1}{2} \int_M (d\varphi \wedge *d\varphi + m^2 \varphi \wedge *\varphi).$$

From the expression of $S_{\mathcal{M}}$, we find the classical stress-energy tensor (written in some coordinate neighborhood) via functional differentiation with respect to the metric:

$$\begin{aligned} T_{ij}^{\mathcal{M}}(p) &= \frac{2}{\sqrt{|\det g_h(p)|}} \frac{\delta S_{\mathcal{M}[h]}}{\delta g_h^{ij}(p)} \Big|_0 \\ &= \nabla_i \varphi(p) \nabla_j \varphi(p) - \frac{1}{2} g_{ij}(p) g^{kl}(p) \nabla_k \varphi(p) \nabla_l \varphi(p) - \frac{1}{2} m^2 g_{ij}(p) \varphi^2(p). \end{aligned}$$

The choice of a quasi-free Hadamard state τ allows us to promote $T_{ij}^{\mathcal{M}}$ to the renormalized quantum stress-energy tensor $\mathcal{T}_{ij}^{\mathcal{M}}$ simply with the formal replacement of the classical field $\varphi(p)$ with (the matrix elements of) the unsmeared field $p \in M \mapsto \langle \eta, \Psi_{\tau}^{\mathcal{M}}(p) \xi \rangle_{\tau}^{\mathcal{M}}$ defined in eq. (3.2.13) for each $\eta, \xi \in \mathcal{V}_{\tau}^{\mathcal{M}}$. This regularization procedure is known as point-splitting (refer to [Wal94, eq. 4.6.5, p. 88]): For each $\eta, \xi \in \mathcal{V}_{\tau}^{\mathcal{M}}$, we choose two “near” points p and q in M and a curve γ connecting them and, parallel transporting along the curve γ , we write

$$\begin{aligned} \langle \eta, \mathcal{T}_{\tau}^{\mathcal{M}ij}(p, q) \xi \rangle_{\tau}^{\mathcal{M}} &= \langle \eta, \nabla^i \Psi_{\tau}^{\mathcal{M}}(p) \nabla^j \Psi_{\tau}^{\mathcal{M}}(q) \xi \rangle_{\tau}^{\mathcal{M}} \\ &\quad - \frac{1}{2} g^{ia}(p) Y_{\gamma a}^j Y_{\gamma b}^{kb}(p) Y_{\gamma l}^l \langle \eta, \nabla_k \Psi_{\tau}^{\mathcal{M}}(p) \nabla_l \Psi_{\tau}^{\mathcal{M}}(q) \xi \rangle_{\tau}^{\mathcal{M}} \\ &\quad - \frac{1}{2} m^2 g^{ia}(p) Y_{\gamma a}^j \langle \eta, \Psi_{\tau}^{\mathcal{M}}(p) \Psi_{\tau}^{\mathcal{M}}(q) \xi \rangle_{\tau}^{\mathcal{M}}. \end{aligned} \quad (3.2.15)$$

Finally we must take the limit $q \rightarrow p$ once that all the divergences are removed. The point-splitting procedure involves the parallel transport Y_{γ} (see Definition 1.1.21), which depends upon the choice of the curve γ connecting the point p to the point q . It follows that the expression above depends on the choice of such curve. Anyway this ambiguity is avoided if we assume that p and q are sufficiently close to have a unique geodesic connecting them and we choose such geodesic as γ . This assumption can be done because in our calculation we will finally take the limit $q \rightarrow p$ along the chosen curve. As a matter of fact the expression of the stress-energy tensor renormalized with respect to the state τ as reference differs from the expression given above by a multiple of the identity operator. However such term is irrelevant for our calculations since the stress-energy tensor will appear only inside a commutator.

As we said, the stress-energy tensor appears in our subsequent calculations only in a commutator, specifically a commutator with an arbitrary Weyl generator (represented via $\pi_{\tau}^{\mathcal{M}}$). A cursory glance to eq. (3.2.15) shows that it is useful for us to evaluate the matrix elements of the commutator of two unsmeared fields with an arbitrary represented Weyl generator. To this end we evaluate separately the matrix elements arising from the LHS and the RHS of eq. (3.2.2). We fix $\eta, \xi \in \mathcal{V}_{\tau}^{\mathcal{M}}$, $f, f' \in \Omega_0^0 M$ and $\varphi \in V$, where $(V, \sigma) = \mathcal{B}(\mathcal{M}, \Lambda^0 M, A)$, and we use eq. (3.2.14)

twice:

$$\begin{aligned} \iint_M \langle \eta, [\Psi_\tau^\mathcal{M}(p) \Psi_\tau^\mathcal{M}(q), V_\tau^\mathcal{M}(\varphi)] \xi \rangle_\tau^\mathcal{M} f(p) f'(q) d\mu_g(p) d\mu_g(q) \\ = \langle \eta, [\Psi_\tau^\mathcal{M}(f) \Psi_\tau^\mathcal{M}(f'), V_\tau^\mathcal{M}(\varphi)] \xi \rangle_\tau^\mathcal{M}. \end{aligned}$$

Now we exploit also the definition of the symplectic form σ (cfr. Lemma 2.2.3):

$$\begin{aligned} -\sigma(e_A f, \varphi) \langle \eta, V_\tau^\mathcal{M}(\varphi) \Psi_\tau^\mathcal{M}(f') \xi \rangle_\tau^\mathcal{M} \\ = \iint_M \varphi(p) f(p) f'(q) \langle \eta, V_\tau^\mathcal{M}(\varphi) \Psi_\tau^\mathcal{M}(q) \xi \rangle_\tau^\mathcal{M} d\mu_g(p) d\mu_g(q), \\ -\sigma(e_A f', \varphi) \langle \eta, \Psi_\tau^\mathcal{M}(f) V_\tau^\mathcal{M}(\varphi) \xi \rangle_\tau^\mathcal{M} \\ = \iint_M \varphi(q) f'(q) f(p) \langle \eta, \Psi_\tau^\mathcal{M}(p) V_\tau^\mathcal{M}(\varphi) \xi \rangle_\tau^\mathcal{M} d\mu_g(p) d\mu_g(q). \end{aligned}$$

From eq. (3.2.2) and the freedom in the choice of f and f' we deduce that

$$\begin{aligned} \langle \eta, [\Psi_\tau^\mathcal{M}(p) \Psi_\tau^\mathcal{M}(q), V_\tau^\mathcal{M}(\varphi)] \xi \rangle_\tau^\mathcal{M} \\ = \varphi(p) \langle \eta, V_\tau^\mathcal{M}(\varphi) \Psi_\tau^\mathcal{M}(q) \xi \rangle_\tau^\mathcal{M} + \varphi(q) \langle \eta, \Psi_\tau^\mathcal{M}(p) V_\tau^\mathcal{M}(\varphi) \xi \rangle_\tau^\mathcal{M} \quad (3.2.16) \end{aligned}$$

for each $\varphi \in V$ and each $\eta, \xi \in \mathcal{V}_\tau^\mathcal{M}$.

Main theorem

We are ready to prove that the action of the functional derivative of the relative Cauchy evolution with respect to the spacetime metric agrees with the action of the quantum stress-energy tensor in the case of the Klein-Gordon field. As a matter of fact the main part of the proof has already been discussed in the previous parts of the current subsection. Here we simply state the theorem and put together all the ingredients.

Theorem 3.2.4. *Let $\mathcal{A} : \mathbf{ghs}^{KG} \rightarrow \mathbf{alg}$ be the locally covariant quantum field theory for the Klein-Gordon field obtained specializing the result of Section 2.2 to the situation of Subsection 2.3.1 and let $(\mathcal{M}, \Lambda^0 M, A)$ be an object of the category \mathbf{ghs}^{KG} defined there. Consider a quasi-free Hadamard state τ on the CCR representation $(\mathcal{V}, V) = \mathcal{A}(\mathcal{M}, \Lambda^0 M, A)$ and denote the GNS triple induced by τ with $(\mathcal{H}_\tau^\mathcal{M}, \pi_\tau^\mathcal{M}, \Omega_\tau^\mathcal{M})$. We denote with $V_\tau^\mathcal{M}$ the represented counterpart of the Weyl map V (cfr. eq. (3.2.10)) and with $\mathcal{T}_\tau^\mathcal{M}$ the quantum stress-energy tensor for the Klein-Gordon field on \mathcal{M} obtained via point-splitting in the representation induced by the state τ (cfr. eq. (3.2.15)). Then there exists a dense subspace $\mathcal{V}_\tau^\mathcal{M}$ of $\mathcal{H}_\tau^\mathcal{M}$ such*

that

$$\frac{\delta}{\delta h} \pi_{\tau}^{\mathcal{M}} (R_h^{\mathcal{M}} (V(\varphi))) = -\frac{\imath}{2} [\mathcal{T}_{\tau}^{\mathcal{M}}, V_{\tau}^{\mathcal{M}} (\varphi)] \quad \forall \varphi \in V$$

in the sense of quadratic forms on $\mathcal{V}_{\tau}^{\mathcal{M}}$.

Proof. A dense subspace $\mathcal{V}_{\tau}^{\mathcal{M}}$ of $\mathcal{H}_{\tau}^{\mathcal{M}}$ exists by virtue of the choice of a quasi-free Hadamard state τ (see few lines before eq. (3.2.12)). The thesis means that

$$\left\langle \theta, \frac{\delta}{\delta h} \pi_{\tau}^{\mathcal{M}} (R_h^{\mathcal{M}} (V(\varphi))) \theta \right\rangle_{\tau}^{\mathcal{M}} = -\frac{\imath}{2} \langle \theta, [\mathcal{T}_{\tau}^{\mathcal{M}}, V_{\tau}^{\mathcal{M}} (\varphi)] \theta \rangle_{\tau}^{\mathcal{M}}$$

for each $\theta \in \mathcal{V}_{\tau}^{\mathcal{M}}$ and each $\varphi \in V$, where $\langle \cdot, \cdot \rangle_{\tau}^{\mathcal{M}}$ denotes the scalar product on the Hilbert space $\mathcal{H}_{\tau}^{\mathcal{M}}$.

We fix a compact subset K of M and 1-parameter family of globally hyperbolic perturbations $(-1, 1) \rightarrow GHP(\mathcal{M}, K)$, $s \mapsto h^s$ such that $h^0 = 0$. Using the definition of $\frac{\delta}{\delta h} R_h^{\mathcal{M}} (V(\varphi))$, we may find an equivalent form of our thesis (we still adopt the notation $\delta_s = \mathrm{d}/\mathrm{d}s|_0$):

$$\delta_s \langle \theta, \pi_{\tau}^{\mathcal{M}} (R_{h^s}^{\mathcal{M}} (V(\varphi))) \theta \rangle_{\tau}^{\mathcal{M}} = -\frac{\imath}{2} \int_M (\delta_s h^s) \left(\langle \theta, [\mathcal{T}_{\tau}^{\mathcal{M}}, V_{\tau}^{\mathcal{M}} (\varphi)] \theta \rangle_{\tau}^{\mathcal{M}} \right) \mathrm{d}\mu_g,$$

where we are considering the dual pairing between $T^*M \otimes_s T^*M$ and $TM \otimes_s TM$ in the integrand appearing on the RHS.

Recall that $R_h^{\mathcal{M}} = \mathcal{C} (r_h^{\mathcal{M}})$ (cfr. eq. (3.2.4)) and the properties of the quantization functor \mathcal{C} defined in Subsection 2.2.2. We deduce that $\pi_{\tau}^{\mathcal{M}} \circ R_{h^s}^{\mathcal{M}} \circ V = \pi_{\tau}^{\mathcal{M}} \circ V \circ r_{h^s}^{\mathcal{M}} = V_{\tau}^{\mathcal{M}} \circ r_{h^s}^{\mathcal{M}}$. This observation entails another slight modification of the thesis:

$$\delta_s \langle \theta, V_{\tau}^{\mathcal{M}} (r_{h^s}^{\mathcal{M}} \varphi) \theta \rangle_{\tau}^{\mathcal{M}} = -\frac{\imath}{2} \int_M (\delta_s h^s) \left(\langle \theta, [\mathcal{T}_{\tau}^{\mathcal{M}}, V_{\tau}^{\mathcal{M}} (\varphi)] \theta \rangle_{\tau}^{\mathcal{M}} \right) \mathrm{d}\mu_g.$$

Now we exploit eq. (3.2.12) and we eliminate the factor $\imath/2$ on both sides of the resulting equation:

$$\langle \theta, \{ \Phi_{\tau}^{\mathcal{M}} (\delta_s r_{h^s}^{\mathcal{M}} \varphi), V_{\tau}^{\mathcal{M}} (\varphi) \} \theta \rangle_{\tau}^{\mathcal{M}} = - \int_M (\delta_s h^s) \left(\langle \theta, [\mathcal{T}_{\tau}^{\mathcal{M}}, V_{\tau}^{\mathcal{M}} (\varphi)] \theta \rangle_{\tau}^{\mathcal{M}} \right) \mathrm{d}\mu_g.$$

We still want to reformulate the thesis a little bit using eq. (3.2.7) and eq. (3.2.11):

$$\underbrace{\langle \theta, \{ \Psi_{\tau}^{\mathcal{M}} (\delta_s A[h^s] \varphi), V_{\tau}^{\mathcal{M}} (\varphi) \} \theta \rangle_{\tau}^{\mathcal{M}}}_{\text{L}} = - \underbrace{\int_M (\delta_s h^s) \left(\langle \theta, [\mathcal{T}_{\tau}^{\mathcal{M}}, V_{\tau}^{\mathcal{M}} (\varphi)] \theta \rangle_{\tau}^{\mathcal{M}} \right) \mathrm{d}\mu_g}_{\text{R}}.$$

Now we work with the LHS of the last equation (denoted by L) and the RHS (denoted by R) separately. Starting from L, we exploit the relation between smeared

and unsmeared fields, eq. (3.2.14):

$$\mathbf{L} = \int_M \langle \theta, \{ \Psi_\tau^{\mathcal{M}}(p), V_\tau^{\mathcal{M}}(\varphi) \} \theta \rangle_\tau^{\mathcal{M}} (\delta_s A[h^s] \varphi)(p) d\mu_g.$$

We want to express \mathbf{L} using oriented coordinate neighborhoods. Indeed we can find an open covering of M constituted by coordinate neighborhoods. In order to make calculations easier, we choose these coordinate neighborhoods in such a way that on each of them $|\det g| = 1$. We can exploit the paracompactness of the manifold M to pick out a locally finite refinement and we introduce a partition of unity subordinate to the refined covering. Since $\text{supp}(h^s) \subseteq K$ for each $s \in (-1, 1)$, the support of the coefficients in $\delta_s A[h^s]$ must be included in K too. Exploiting the compactness of K , we can find that only a finite number of the coordinate neighborhoods considered so far intersect it. We denote them with $\{(U_\alpha, V_\alpha, \phi_\alpha)\}$ and we consider only the corresponding members $\{\chi_\alpha\}$ in the partition of unity (the other members indeed have null product with the integrand). This entails that we can use this finite collection of coordinate neighborhoods (together with the corresponding members of the original partition of unity) to express \mathbf{L} in local coordinates:

$$\mathbf{L} = \sum_\alpha \int_{V_\alpha} \chi_\alpha \langle \theta, \{ \Psi_\tau^{\mathcal{M}}(x), V_\tau^{\mathcal{M}}(\varphi) \} \theta \rangle_\tau^{\mathcal{M}} (\delta_s A[h^s] \varphi)(x) dV,$$

where dV denotes the standard volume form on \mathbb{R}^4 and all the sections that appear inside the integral are now written in local coordinates⁴. It is convenient to define

$$\begin{aligned} \zeta : M &\rightarrow \mathbb{C} \\ p &\mapsto \langle \theta, \{ \Psi_\tau^{\mathcal{M}}(x), V_\tau^{\mathcal{M}}(\varphi) \} \theta \rangle_\tau^{\mathcal{M}} \end{aligned}$$

in order to simplify our notation. Now we use eq. (3.2.8). In this way we obtain

$$\mathbf{L} = \underbrace{\sum_\alpha \int_{V_\alpha} \chi_\alpha \zeta (\nabla^i \nabla^j \varphi) \delta_s h_{ij}^s dV}_{\mathbf{L}_1} + \underbrace{\sum_\alpha \int_{V_\alpha} \chi_\alpha \zeta (\nabla_k \varphi) \delta_s \Gamma[h^s]_{ij}^k g^{ij} dV}_{\mathbf{L}_2},$$

where the dependence of the integrand on the point $x \in V_\alpha$ now is understood. We denote the first addend appearing on the RHS of the last equation with \mathbf{L}_1 and the second with \mathbf{L}_2 . We integrate \mathbf{L}_1 by parts noting that χ_α is null on the boundary of

⁴by this we mean that, inside the integral over V_α , $\delta_s A[h^s] \varphi$ now denotes the push-forward through ϕ_α of the original $\delta_s A[h^s] \varphi$ restricted to U_α and similarly for the other sections inside the integral

V_α , hence no surface term appears:

$$\begin{aligned} L_1 = & \underbrace{- \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} (\nabla^i \zeta) (\nabla^j \varphi) \delta_s h_{ij}^s dV}_{\text{X}} - \underbrace{\sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta (\nabla^j \varphi) \nabla^i \delta_s h_{ij}^s dV}_{L_3} \\ & - \underbrace{\sum_{\alpha} \int_{V_{\alpha}} (\nabla^i \chi_{\alpha}) \zeta (\nabla^j \varphi) \delta_s h_{ij}^s dV}_{=0}. \end{aligned}$$

The last term in the equation above gives null contribution. We can check this fact observing that, on each point of the support of $\delta_s h^s$, the finite number of χ_{α} sum up to 1, hence their derivatives sum up to zero. We denote the first of the remaining terms with X and the second with L_3 . Up to now we have

$$L = X + L_2 + L_3.$$

Now we investigate R. As we did for L, we express it using the chosen local coordinates:

$$R = - \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha}(x) (\delta_s h_{ij}^s)(x) \langle \theta, [\mathcal{T}_{\tau}^{\mathcal{M}ij}(x), V_{\tau}^{\mathcal{M}}(\varphi)] \theta \rangle_{\tau}^{\mathcal{M}} dV.$$

Consider the integrand (dropping χ_{α} for the moment). Inside the commutator appears the quantized stress-energy tensor. Indeed we have eq. (3.2.15) that tells us about its form, but we must perform the coincidence limit before we can insert such equation inside the integral in place of $\mathcal{T}_{\tau}^{\mathcal{M}}$. As a matter of fact we previously calculate the expectation value of the commutator recalling the commutation relation found in eq. (3.2.16) and only after that we take the coincidence limit as required by the point-splitting procedure realizing that no divergences arise. Exploiting also the symmetry of $\delta_s h^s$ and g , anticommutators appear. All these operations produce the following result (to shorten the expression we replace $\langle \theta, \{\Psi_{\tau}^{\mathcal{M}}(x), V_{\tau}^{\mathcal{M}}(\varphi)\} \theta \rangle_{\tau}^{\mathcal{M}}$ with ζ as above):

$$\begin{aligned} R = & \underbrace{- \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} (\nabla^i \varphi) (\nabla^j \zeta) \delta_s h_{ij}^s dV}_{=X} \\ & + \underbrace{\frac{1}{2} \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} g^{kl} (\nabla_k \varphi) (\nabla_l \zeta) g^{ij} \delta_s h_{ij}^s dV}_{R_1} + \underbrace{\frac{1}{2} m^2 \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \varphi \zeta g^{ij} \delta_s h_{ij}^s dV}_{R_2}. \end{aligned}$$

The first term coincides with the term X in L_1 once that the indices i and j are interchanged taking into account the symmetry of $\delta_s h^s$. As for the other two terms, some more work is required. We denote the first one with R_1 and the second one

with R_2 and we integrate R_1 by parts (this time we directly omit the term containing derivatives of χ_α since it gives null contribution as noted above):

$$R_1 = -\frac{1}{2} \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} g^{kl} (\nabla_l \nabla_k \varphi) \zeta g^{ij} \delta_s h_{ij}^s dV - \frac{1}{2} \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} g^{kl} (\nabla_k \varphi) \zeta g^{ij} \nabla_l \delta_s h_{ij}^s dV.$$

If we put together R_1 and R_2 and we remind that $A\varphi = 0$ since $\varphi \in V$, we get

$$\begin{aligned} R_1 + R_2 &= \frac{1}{2} \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \overbrace{(-g^{kl} \nabla_l \nabla_k \varphi + m^2 \varphi)}^{=A\varphi=0} \zeta g^{ij} \delta h_{ij}^s dV \\ &\quad - \frac{1}{2} \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} g^{kl} (\nabla_k \varphi) \zeta g^{ij} \nabla_l \delta h_{ij}^s dV \\ &= -\frac{1}{2} \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta (\nabla_k \varphi) g^{ij} g^{kl} \nabla_l \delta h_{ij}^s dV. \end{aligned}$$

At this stage our thesis $L = R$ is reduced to the following identity:

$$L_2 + L_3 = R_1 + R_2. \quad (3.2.17)$$

The next step consist in the proof of the identity

$$\delta_s \Gamma [h^s]_{ij}^k g^{ij} - g^{jk} \nabla^i \delta_s h_{ij}^s = -\frac{1}{2} g^{ij} g^{kl} \nabla_l \delta_s h_{ij}^s \quad (3.2.18)$$

in each point of M . If this identity actually holds everywhere, it follows that eq. (3.2.17) holds too and hence the proof is complete: In fact, as the reader might easily check, eq. (3.2.18) written using the coordinate neighborhoods $(U_{\alpha}, V_{\alpha}, \phi_{\alpha})$, integrated on both sides on each V_{α} together with the factor $\chi_{\alpha} \zeta \nabla_k \varphi$ and summed over the finite number of indices α gives exactly eq. (3.2.17).

The first thing we do is to use the metric to lower the index on ∇ in the second term on the LHS of eq. (3.2.18) and, after that, we rename some summation indices (bear in mind that g and $\delta_s h^s$ are symmetric). In this way the identity in eq. (3.2.18) to be checked becomes:

$$\delta_s \Gamma [h^s]_{ij}^k g^{ij} - g^{ij} g^{lk} \nabla_i \delta_s h_{lj}^s = -\frac{1}{2} g^{ij} g^{kl} \nabla_l \delta_s h_{ij}^s. \quad (3.2.19)$$

Now we fix an arbitrary point p in M and we choose Riemannian normal coordinates in a (sufficiently small) neighborhood of p (cfr. e.g. [Wal84, Sect. 3.3, p. 42]). Doing so, we put ourselves in a favorable situation from a computational point of view since with this choice of coordinates the Christoffel symbols Γ_{ij}^k are null at p and hence we can freely replace ∇ with ∂ (note that a similar result does not hold for the Christoffel symbols of a “perturbed” connection $\nabla [h^s]$). With this considerations we evaluate $\delta_s \Gamma [h^s]_{ij}^k$ (recall eq. (1.1.1) which provides the expression of the Christoffel symbols

for the Levi-Civita connection):

$$\begin{aligned} \delta_s \Gamma [h^s]_{ij}^k &= (\delta_s g_{h^s}^{kl}) \overbrace{\frac{1}{2} (\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij})}^{=g_{lm} \Gamma_{ij}^m=0} + g^{kl} \frac{1}{2} (\partial_i \delta_s h_{lj}^s + \partial_j \delta_s h_{il}^s - \partial_l \delta_s h_{ij}^s) \\ &= g^{kl} \frac{1}{2} (\partial_i \delta_s h_{lj}^s + \partial_j \delta_s h_{il}^s - \partial_l \delta_s h_{ij}^s). \end{aligned} \quad (3.2.20)$$

As a matter of fact we are interested in the contraction of $\delta_s \Gamma [h^s]_{ij}^k$ with g^{ij} :

$$\delta_s \Gamma [h^s]_{ij}^k g^{ij} = g^{ij} g^{kl} \partial_i \delta_s h_{lj}^s - \frac{1}{2} g^{ij} g^{kl} \partial_l \delta_s h_{ij}^s, \quad (3.2.21)$$

where we exploited the identity $(\partial_j \delta_s h_{il}^s) g^{ij} = (\partial_i \delta_s h_{jl}^s) g^{ij}$. With this result we evaluate the LHS of eq. (3.2.19):

$$\begin{aligned} \delta_s \Gamma [h^s]_{ij}^k g^{ij} - g^{ij} g^{lk} \nabla_i \delta_s h_{lj}^s &= g^{ij} g^{kl} \partial_i \delta_s h_{lj}^s - \frac{1}{2} g^{ij} g^{kl} \partial_l \delta_s h_{ij}^s - g^{ij} g^{lk} \partial_i \delta_s h_{lj}^s \\ &= -\frac{1}{2} g^{ij} g^{kl} \partial_l \delta_s h_{ij}^s. \end{aligned}$$

It is sufficient to restore ∇ in place of ∂ on the RHS of the last equation to realize that eq. (3.2.19) actually is proved. We already showed that this one is equivalent to eq. (3.2.18), which in turn entails (3.2.17). This completes the proof. \square

3.2.3 Relative Cauchy evolution for the Proca field

Now we turn our attention to the Proca field. Our aim is to extend the result obtained for the Klein-Gordon field also in the present context, that is to prove the agreement of the action of the functional derivative of the relative Cauchy evolution with the action of the quantized stress-energy tensor for the Proca field.

We need some preparation also in this case. We will follow an approach very similar to that of the previous subsection. The main difference lies in the fact that now we are going to take into account the results of Subsection 2.3.2 in place of those from Subsection 2.3.1, specifically we consider the locally covariant quantum field theory $\mathcal{A} : \mathbf{ghs}^P \rightarrow \mathbf{alg}$ (cfr. Definition 2.3.8 for the definition of the category) defined as the composition of the covariant functor $\mathcal{B} : \mathbf{ghs}^P \rightarrow \mathbf{ssp}$ describing the classical theory of the Proca field (see Theorem 2.3.10) with the usual quantization functor $\mathcal{C} : \mathbf{ssp} \rightarrow \mathbf{alg}$ (see Lemma 2.2.7). After the proof of Theorem 2.3.10 we argued that \mathcal{A} fulfils the time slice axiom as a LCQFT (indeed the causality condition holds too, but this fact is not relevant in this context). This ensures that one can actually consider the RCE for the Proca field as presented in Section 3.1 and all the results found there still hold since now we are only considering a richer structure on each spacetime, but the morphisms considered there are easily recognized to induce morphisms also in this context.

From now on we use the notation of Subsection 2.3.2. In particular we recall that the differential operators considered here (which are formally selfadjoint, but fail to be normally hyperbolic) are of the form

$$A = \delta d + m^2 \text{id}_{\Omega^1 M} : \Omega^1 M \rightarrow \Omega^1 M$$

on each globally hyperbolic spacetime $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$. At the same time we also consider a formally selfadjoint normally hyperbolic operator

$$P_A = \square_1 + m^2 \text{id}_{\Omega^1 M} : \Omega^1 M \rightarrow \Omega^1 M.$$

We denote with $e_A^{a/r}$ its associated advanced/retarded Green operator and we use it to define the advanced/retarded Green operator for A (cfr. Lemma 2.3.6):

$$f_A^{a/r} = e_P^{a/r} \circ \left(\text{id}_{\Omega_0^1 M} + \frac{1}{m^2} d\delta \right) : \Omega_0^1 M \rightarrow \Omega^1 M.$$

Relative Cauchy evolution for the classical Proca field

Our first purpose is to find a convenient expression for the relative Cauchy evolution of the Proca field at a classical level and to relate it to the original RCE. To do this we need a result similar to Proposition 3.2.3. Note that the object $(\mathcal{M}|_O, \Lambda^1 O, A|_O)$ of \mathfrak{ghs}^P that we are going to take into account is defined exactly with the procedure followed for the corresponding object of \mathfrak{ghs}^{KG} with the only replacement of Λ^0 with Λ^1 (see the first part of the proof in Proposition 3.2.3).

Proposition 3.2.5. *Let $\mathcal{B} : \mathfrak{ghs}^P \rightarrow \mathfrak{ssp}$ be the covariant functor describing the classical theory of the Proca field (cfr. Subsection 2.3.2), consider an object $(\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t}), \Lambda^1 M, A)$ of \mathfrak{ghs}^P and let O be an \mathcal{M} -causally convex connected open subset of M including a smooth spacelike Cauchy surface Σ for \mathcal{M} . Consider the object $(\mathcal{M}|_O, \Lambda^1 O, A|_O)$ of \mathfrak{ghs}^P and the morphism $(\iota_O^M, \iota_{\Lambda^1 O}^{\Lambda^1 M})$ of \mathfrak{ghs}^P from $(\mathcal{M}|_O, \Lambda^1 O, A|_O)$ to $(\mathcal{M}, \Lambda^1 M, A)$ induced by the inclusion maps $\iota_O^M : O \rightarrow M$ and $\iota_{\Lambda^1 O}^{\Lambda^1 M} : \Lambda^1 O \rightarrow \Lambda^1 M$. Then there exists a partition of unity $\{\chi^a, \chi^r\}$ on M such that the inverse $\mathcal{B}(\iota_O^M, \iota_{\Lambda^1 O}^{\Lambda^1 M})^{-1}$ of the bijective morphism $\mathcal{B}(\iota_O^M, \iota_{\Lambda^1 O}^{\Lambda^1 M})$ of \mathfrak{ssp} from $(V, \sigma) = \mathcal{B}(\mathcal{M}|_O, \Lambda^1 O, A|_O)$ to $(W, \omega) = \mathcal{B}(\mathcal{M}, \Lambda^1 M, A)$ satisfies the following equation:*

$$\mathcal{B}(\iota_O^M, \iota_{\Lambda^1 O}^{\Lambda^1 M})^{-1} \Theta = \pm f_{A|_O} \left(\text{res}_{\iota_{\Lambda^1 O}^{\Lambda^1 M}} (A(\chi^{a/r} \Theta)) \right) \quad \forall \Theta \in W,$$

where $f_{A|_O}$ is the causal propagator for $A|_O$ and the restriction map is defined in Lemma 2.2.4.

Proof. The most part of this proof is identical to the proof of Proposition 3.2.3, provided that you replace everywhere Λ^0 , Ω^0 , φ , $A = \square_0 + m^2 \text{id}_{\Omega^0 M}$ and its causal

propagator e_A with Λ^1 , Ω^1 , Θ , the current linear differential operator $A = \delta d + m^2 \text{id}_{\Omega^1 M}$ and its causal propagator f_A (whose existence follows from Lemma 2.3.6). You should also remember that in the present situation there is a (potentially) stricter condition on the morphisms of \mathbf{ghs}^P , that is compatibility with both δd and $d\delta$, but this does not give rise to problems of any sort because the inclusion maps easily satisfy this requirement. The time slice axiom holds also in this situation as we proved in Theorem 2.3.10, hence $\mathcal{B}(\iota_O^M, \iota_{\Lambda^1 O}^{\Lambda^1 M})$ is bijective and its inverse $\mathcal{B}(\iota_O^M, \iota_{\Lambda^1 O}^{\Lambda^1 M})^{-1}$ is a morphism of \mathbf{ssp} . Our aim is to find a convenient expression for $\mathcal{B}(\iota_O^M, \iota_{\Lambda^1 O}^{\Lambda^1 M})^{-1}$. The only slight difference arises when we check the identity in the statement. To be precise, we obtain the next equation following exactly the same reasoning:

$$\mathcal{B}(\iota_O^M, \iota_{\Lambda^1 O}^{\Lambda^1 M})(\alpha\Theta) = \pm f_A A(\chi^{a/r}\Theta),$$

where α denotes the map from W to V defined by

$$\alpha\Theta = \text{res}_{\iota_{\Lambda^1 O}^{\Lambda^1 M}}(A(\chi^{a/r}\Theta))$$

for each $\Theta \in W$ and $\chi^{a/r}$ is the partition of unity that we find imitating the first part of the proof of Proposition 3.2.3. Now we would like to apply Lemma 1.3.17, but this cannot be done directly since no normally hyperbolic operator is immediately available. Anyway this problem is easily circumvented recalling that

$$f_A = e_A \circ \left(\text{id}_{\Omega_0^1 M} + \frac{1}{m^2} d\delta \right)$$

and that

$$\left(\text{id}_{\Omega^1 M} + \frac{1}{m^2} d\delta \right) \circ A = P_A.$$

With these observations we find

$$\mathcal{B}(\iota_O^M, \iota_{\Lambda^1 O}^{\Lambda^1 M})(\alpha\Theta) = \pm e_A P_A \Theta^{a/r},$$

where $\Theta^{a/r} = \chi^{a/r}\Theta$. Now a normally hyperbolic operator P_A is available, but we need to show that $P_A \Theta^{a/r}$ has compact support in order to exploit Lemma 1.3.17. This can be done easily because $A\Theta = 0$ trivially entails $\delta\Theta = 0$; therefore we have $\delta\Theta^a = -\delta\Theta^r$. From this identity we deduce that $\delta\Theta^a$ has compact support (the proof is based on the support properties of the causal propagator f_A and of the partition of unity). Since $P_A \Theta^a = A\Theta^a + d\delta\Theta^a = -P_A \Theta^r$, we can conclude that $P_A \Theta^a$ actually has compact support and hence we are allowed to apply Lemma 1.3.17 obtaining

$$\pm e_A P_A \Theta^{a/r} = \pm (e_A^a P_A \Theta^{a/r} - e_A^r P_A \Theta^{a/r}) = \Theta^a + \Theta^r = \Theta.$$

With this we conclude

$$\mathcal{B} \left(\iota_O^M, \iota_{\Lambda^1 O}^{\Lambda^1 M} \right) (\alpha \Theta) = \Theta = \mathcal{B} \left(\iota_O^M, \iota_{\Lambda^1 O}^{\Lambda^1 M} \right) \left(\mathcal{B} \left(\iota_O^M, \iota_{\Lambda^1 O}^{\Lambda^1 M} \right)^{-1} \Theta \right) \quad \forall \Theta \in W.$$

Since $\mathcal{B} \left(\iota_O^M, \iota_{\Lambda^1 O}^{\Lambda^1 M} \right)$ is injective, the last equation entails

$$\mathcal{B} \left(\iota_O^M, \iota_{\Lambda^0 O}^{\Lambda^0 M} \right)^{-1} \Theta = \alpha \Theta \quad \forall \Theta \in W,$$

which is exactly our thesis. \square

As we did in the case of the Klein-Gordon field, we specialize the definition of the RCE to the current situation. Consider an object $(\mathcal{M}, \Lambda^1 M, A)$ of \mathbf{ghs}^P , take $h \in GHP(\mathcal{M})$ and recall the definitions of the morphisms $i_{\pm}^{\mathcal{M}}[h]$ and $j_{\pm}^{\mathcal{M}}[h]$ introduced before Definition 3.1.3. Together with the perturbed spacetime $\mathcal{M}[h]$, we must also consider the effects of h on the inner product defined on the vector bundle $\Lambda^1 M$ and on the differential operator $A = \delta d + m^2 \text{id}_{\Omega^1 M}$. The inner product on $\Lambda^1 M$ is induced by the metric, hence we should consider the inner product induced by the perturbed metric g_h . As for the linear differential operator we define $A[h] = \delta[h] d + m^2 \text{id}_{\Omega^1 M}$, where $\delta[h]$ is the codifferential over $\mathcal{M}[h]$. Similarly we have to consider $P_A[h] = \square_1[h] + m^2 \text{id}_{\Omega^1 M}$, where $\square_1[h] = d\delta[h] + \delta[h]d$ is the d'Alembert operator defined over $\mathcal{M}[h]$ for 1-forms. As a matter of fact we are replacing the metric g with $g_h = g + h$ whenever there is something related to the metric. We may consider the inclusion map $\iota_{\Lambda^1 M_{\pm}}^{\Lambda^1 M}$, where $M_{\pm} = M \setminus J_{\mp}^{\mathcal{M}}(\text{supp}(h))$ in accordance with the definitions of $i_{\pm}^{\mathcal{M}}[h]$ and $j_{\pm}^{\mathcal{M}}[h]$. Compatibility with both δd and $d\delta$ via $\left(\iota_{M_{\pm}}^M, \iota_{\Lambda^1 M_{\pm}}^{\Lambda^1 M} \right)$ holds (cfr. Definition 2.3.8). Since the effects of the perturbation h are relevant only inside $\text{supp}(h)$, we realize that $\delta[h]$ and δ act exactly in the same way on sections supported outside $\text{supp}(h)$. Together with $A|_{M_{\pm}}$, we may consider $A[h]|_{M_{\pm}}$ and we immediately recognize that they coincide (we denote both of them with $A_{\pm}[h]$). Similarly $P_A|_{M_{\pm}} = P_A[h]|_{M_{\pm}}$ so that we denote both with $P_{A\pm}[h]$. Hence we can introduce the objects $(\mathcal{M}[h], \Lambda^1 M, A[h])$ and $(\mathcal{M}_{\pm}[h], \Lambda^1 M_{\pm}, A_{\pm}[h])$ of \mathbf{ghs}^P and interpret the vector bundle homomorphism $\left(\iota_{M_{\pm}}^M, \iota_{\Lambda^0 M_{\pm}}^{\Lambda^0 M} \right) : \Lambda^1 M_{\pm} \rightarrow \Lambda^1 M$ in the following ways:

$$\begin{aligned} \left(i_{\pm}^{\mathcal{M}}[h], i_{\pm}^{\mathcal{M}, \Lambda^1}[h] \right) &\in \text{Mor}_{\mathbf{ghs}^P} \left((\mathcal{M}_{\pm}[h], \Lambda^1 M_{\pm}, A_{\pm}[h]), (\mathcal{M}, \Lambda^1 M, A) \right), \\ \left(j_{\pm}^{\mathcal{M}}[h], j_{\pm}^{\mathcal{M}, \Lambda^1}[h] \right) &\in \text{Mor}_{\mathbf{ghs}^P} \left((\mathcal{M}_{\pm}[h], \Lambda^1 M_{\pm}, A_{\pm}[h]), (\mathcal{M}[h], \Lambda^1 M, A[h]) \right). \end{aligned}$$

Denote with \mathcal{A} the LCQFT (fulfilling both the causality condition and the time slice axiom) built following the procedure of Subsection 2.3.2. For $(\mathcal{M}, \Lambda^1 M, A) \in \text{Obj}_{\mathbf{ghs}^P}$

and $h \in GHP(\mathcal{M})$ we define the RCE for the Proca field as

$$\begin{aligned} R_h^{\mathcal{M}} &= \mathcal{A} \left(i_-^{\mathcal{M}}[h], i_-^{\mathcal{M}, \Lambda^1}[h] \right) \circ \mathcal{A} \left(j_-^{\mathcal{M}}[h], j_-^{\mathcal{M}, \Lambda^1}[h] \right)^{-1} \\ &\quad \circ \mathcal{A} \left(j_+^{\mathcal{M}}[h], j_+^{\mathcal{M}, \Lambda^1}[h] \right) \circ \mathcal{A} \left(i_+^{\mathcal{M}}[h], i_+^{\mathcal{M}, \Lambda^1}[h] \right)^{-1}. \end{aligned}$$

In a similar way one can consider a classical version of the RCE based on the covariant functor $\mathcal{B} : \mathbf{ghs}^P \rightarrow \mathbf{ssp}$ describing the classical theory of the Proca field (this is actually possible due to version of the time slice axiom satisfied by \mathcal{B} , cfr. Theorem 2.3.10):

$$\begin{aligned} r_h^{\mathcal{M}} &= \mathcal{B} \left(i_-^{\mathcal{M}}[h], i_-^{\mathcal{M}, \Lambda^1}[h] \right) \circ \mathcal{B} \left(j_-^{\mathcal{M}}[h], j_-^{\mathcal{M}, \Lambda^1}[h] \right)^{-1} \\ &\quad \circ \mathcal{B} \left(j_+^{\mathcal{M}}[h], j_+^{\mathcal{M}, \Lambda^1}[h] \right) \circ \mathcal{B} \left(i_+^{\mathcal{M}}[h], i_+^{\mathcal{M}, \Lambda^1}[h] \right)^{-1}. \end{aligned}$$

Since the LCQFT \mathcal{A} is obtained via composition of \mathcal{B} with the quantization functor \mathcal{C} presented in Subsection 2.2.2, we realize that

$$R_h^{\mathcal{M}} = \mathcal{C} \left(r_h^{\mathcal{M}} \right). \quad (3.2.22)$$

We can determine the action of $r_h^{\mathcal{M}}$ applying Proposition 3.2.5 and Proposition 2.3.9. To be precise, we find proper partitions of unity $\{\chi_+^a, \chi_+^r\}$ and $\{\chi_-^a, \chi_-^r\}$ on M such that we can express the action of $\mathcal{B} \left(i_+^{\mathcal{M}}[h], i_+^{\mathcal{M}, \Lambda^1}[h] \right)^{-1}$ and respectively of $\mathcal{B} \left(j_-^{\mathcal{M}}[h], j_-^{\mathcal{M}, \Lambda^1}[h] \right)^{-1}$ according to Proposition 3.2.5. If we take $\Theta \in \mathcal{B}(\mathcal{M}, \Lambda^1 M, A)$ and evaluate $r_h^{\mathcal{M}} \Theta$, we easily find the following result:

$$r_h^{\mathcal{M}} \Theta = f_A A[h] \left(\chi_-^{a/r} f_{A[h]} A \left(\chi_+^{a/r} \Theta \right) \right).$$

In the following we will need the expression of $\frac{d}{ds} r_{h^s}^{\mathcal{M}} \Theta \Big|_0$ for an arbitrary smooth 1-parameter family of perturbations of the metric $s \mapsto h^s$. For convenience in the upcoming calculation we write δ_s in place of $\frac{d}{ds} (\cdot) \Big|_0$. Fix now $\Theta \in \mathcal{B}(\mathcal{M}, \Lambda^1 M, A)$, a compact subset K of M and a smooth 1-parameter family of globally hyperbolic perturbations $(-1, 1) \rightarrow GHP(\mathcal{M}, K)$, $s \mapsto h^s$ and evaluate $\delta_s r_{h^s}^{\mathcal{M}} \Theta$. We carry on such calculation with a procedure identical to the one followed for the Klein-Gordon field. We must only pay attention to the application of Lemma 1.3.17, which cannot be exploited directly. For example, if we are dealing with a section Θ in $\Lambda^1 M$ with \mathcal{M} -past/future compact support such that $A\Theta$ has compact support, we must show that also $P_A \Theta$ has compact support and then we can use Lemma 1.3.17 to conclude

$$f_A^{a/r} A\Theta = e_A^{a/r} P_A \Theta = \Theta.$$

In this way we obtain

$$\left. \frac{d}{ds} r_{h^s}^{\mathcal{M}} \Theta \right|_0 = f_A \left(\left. \frac{d}{ds} A[h^s] \right|_0 \right) \Theta. \quad (3.2.23)$$

We are left with the problem of the expression for $\delta_s A[h^s] \Theta$. We know that $A[h^s] = \delta[h^s] d + m^2 \text{id}_{\Omega^1 M}$, where $\delta[h^s]$ denotes the codifferential built with the perturbed metric g_{h^s} . Indeed the term $m^2 \Theta$ gives null contribution to $\delta_s A[h^s] \Theta$, hence we are interested in the evaluation of $\delta_s \delta[h^s] d\Theta$. Using an arbitrary coordinate neighborhood, one can check that

$$\begin{aligned} (\delta[h^s] d\Theta)_k &= g_{h^s}^{ij} \nabla[h^s]_i \left(-\nabla[h^s]_j \Theta_k + \nabla[h^s]_k \Theta_j \right) \\ &= -g_{h^s}^{ij} \partial_i \Pi_{jk} + g_{h^s}^{ij} \Gamma[h^s]_{ij}^l \Pi_{lk} + g_{h^s}^{ij} \Gamma[h^s]_{ik}^l \Pi_{jl}, \end{aligned}$$

where $\Gamma[h^s]_{ij}^k$ are the Christoffel symbols of the Levi-Civita connection $\nabla[h^s]$ on $\mathcal{M}[h^s]$ and

$$\Pi_{ij} = \nabla[h^s]_i \Theta_j - \nabla[h^s]_j \Theta_i = \partial_i \Theta_j - \partial_j \Theta_i = \nabla_i \Theta_j - \nabla_j \Theta_i. \quad (3.2.24)$$

Therefore

$$\begin{aligned} \left. \frac{d}{ds} (A[h^s] \Theta)_k \right|_0 &= \left. \frac{d}{ds} (\delta[h^s] d\Theta)_k \right|_0 \\ &= \left. \frac{d}{ds} h_{ij}^s \right|_0 \nabla^i \Pi_{jk} + \left. \frac{d}{ds} \Gamma[h^s]_{ij}^l \right|_0 g^{ij} \Pi_{lk} + \left. \frac{d}{ds} \Gamma[h^s]_{ik}^l \right|_0 g^{ij} \Pi_{jl}, \end{aligned} \quad (3.2.25)$$

where in the last step we exploited also eq. (3.2.9).

Properties of the GNS representation induced by a quasi-free Hadamard state for the Proca field

We go on imitating what we have already done in the case of the Klein-Gordon field. So we choose a quasi-free Hadamard state τ for the unital C^* -algebra $(\mathcal{V}, V) = \mathcal{A}(\mathcal{M}, \Lambda^1 M, A)$ (which is actually a CCR representation) describing the quantum theory of the Proca field on the globally hyperbolic spacetime \mathcal{M} . With this choice, we introduce the (unique up to unitary equivalence) GNS triple $(\mathcal{H}_\tau^{\mathcal{M}}, \pi_\tau^{\mathcal{M}}, \Omega_\tau^{\mathcal{M}})$ induced by τ and we follow the discussion made in Subsection 3.2.1. In this way we obtain the represented version of the Weyl map V associated to the CCR representation (\mathcal{V}, V) :

$$V_\tau^{\mathcal{M}} = \pi_\tau^{\mathcal{M}} \circ V : V \rightarrow \mathcal{B}(\mathcal{H}_\tau^{\mathcal{M}}), \quad (3.2.26)$$

where $(V, \sigma) = \mathcal{B}(\mathcal{M}, \Lambda^1 M, A)$ is the symplectic space provided by the covariant functor $\mathcal{B} : \mathbf{ghs}^P \rightarrow \mathbf{ssp}$ describing the classical theory of the Proca field. We find

a map

$$\begin{aligned}\Phi_\tau^\mathcal{M} : V &\rightarrow \mathcal{B}(\mathcal{H}_\tau^\mathcal{M}) \\ \Theta &\mapsto \Phi_\tau^\mathcal{M}(\Theta).\end{aligned}$$

satisfying $e^{i\Phi_\tau^\mathcal{M}(\Theta)} = V_\tau^\mathcal{M}(\Theta)$ for each $\Theta \in V$, where $\Phi_\tau^\mathcal{M}(\Theta)$ is selfadjoint. Together with this map, we have the smeared fields (by virtue of the choice of a Hadamard state):

$$\begin{aligned}\Psi_\tau^\mathcal{M} : \Omega_0^1 M &\rightarrow \mathcal{B}(\mathcal{H}_\tau^\mathcal{M}) \\ \theta &\mapsto -i \frac{d}{dt} V_\tau^\mathcal{M}(t f_A \theta) \Big|_0.\end{aligned}$$

As for the general case, it holds that

$$\Psi_\tau^\mathcal{M}(\theta) = \Phi_\tau^\mathcal{M}(f_A \theta) \quad (3.2.27)$$

for each $\theta \in \Omega_0^1 M$ and we recognize $\Psi_\tau^\mathcal{M}$ to be linear.

As we said in Subsection 3.2.1, the choice of a quasi-free Hadamard state τ assures that Assumption 3.1.5 is satisfied, i.e. we are able to find a dense subspace $\mathcal{V}_\tau^\mathcal{M}$ of $\mathcal{H}_\tau^\mathcal{M}$ and a dense sub-*algebra $\mathcal{B}_\tau^\mathcal{M}$ of $\mathcal{A}(\mathcal{M}, \Lambda^1 M, A)$ such that the functional derivative of the RCE with respect to the spacetime metric can be defined. We also have a version of eq. (3.2.3) fitted to the Proca field: for each $\xi \in \mathcal{V}_\tau^\mathcal{M}$, each $\Theta \in V$, each compact subset K of M and each smooth 1-parameter family $(-1, 1) \rightarrow GHP(\mathcal{M}, K)$, $s \mapsto h^s$ such that $h^0 = 0$, it holds that

$$\frac{d}{ds} \langle \xi, V_\tau^\mathcal{M}(r_{h^s}^\mathcal{M} \Theta) \xi \rangle_\tau^\mathcal{M} \Big|_0 = \frac{i}{2} \left\langle \xi, \left\{ \Phi_\tau^\mathcal{M} \left(\frac{d}{ds} (r_{h^s}^\mathcal{M} \Theta) \Big|_0 \right), V_\tau^\mathcal{M}(\Theta) \right\} \xi \right\rangle_\tau^\mathcal{M}. \quad (3.2.28)$$

Moreover one can show that for each $\eta, \xi \in \mathcal{V}_\tau^\mathcal{M}$ there exists a smooth section, denoted by

$$\begin{aligned}M &\rightarrow T_\mathbb{C} M \\ p &\mapsto \langle \eta, \Psi_\tau^\mathcal{M}(p) \xi \rangle_\tau^\mathcal{M},\end{aligned}$$

where $T_\mathbb{C} M$ stands for the complex vector bundle obtained via the tensor product of each fiber of TM with \mathbb{C} and $\langle \cdot, \cdot \rangle_\tau^\mathcal{M}$ denotes the scalar product of the Hilbert space $\mathcal{H}_\tau^\mathcal{M}$, such that

$$\langle \eta, \Psi_\tau^\mathcal{M}(\theta) \xi \rangle_\tau^\mathcal{M} = \int_M (\theta(p)) \left(\langle \eta, \Psi_\tau^\mathcal{M}(p) \xi \rangle_\tau^\mathcal{M} \right) d\mu_g \quad (3.2.29)$$

for each $\theta \in \Omega_0^1 M$, where $d\mu_g$ is the standard volume form on \mathcal{M} and the dual

pairing between T^*M and TM has been taken into account (note that one may indeed write the integrand in the abstract index notation putting a contravariant index on the new smooth section and a covariant index on the test function). We may regard this section as the matrix element of the (unique) unsmeared Proca field induced by the quasi-free Hadamard state τ on the globally hyperbolic spacetime \mathcal{M} .

Quantized stress-energy tensor for the Proca field

We try to define the quantized stress-energy tensor associated to the Proca field through the point-splitting procedure starting from the expression of the classical stress-energy tensor. To obtain it, we need the expression for the action of the Proca field on the globally hyperbolic spacetime \mathcal{M} , which in turn comes from the differential operator $A = \delta d + m^2 \text{id}_{\Omega^1 M}$ governing the classical dynamics of the field:

$$\begin{aligned} S_{\mathcal{M}} &= \frac{1}{2} (\Theta, A\Theta)_{g,1} = \frac{1}{2} (d\Theta, d\Theta)_{g,2} + \frac{1}{2} m^2 (\Theta, \Theta)_{g,1} \\ &= \frac{1}{2} \int_M (d\Theta \wedge *d\Theta + m^2 \Theta \wedge *\Theta). \end{aligned}$$

Taking the functional derivative of $S_{\mathcal{M}}$ with respect to the metric, we find the classical stress-energy tensor for the Proca field (which we express in local coordinates):

$$\begin{aligned} T_{ij}^{\mathcal{M}}(p) &= \left. \frac{2}{\sqrt{|\det g_h(p)|}} \frac{\delta S_{\mathcal{M}}[h]}{\delta g_h^{ij}(p)} \right|_0 \\ &= g^{bd}(p) \Pi_{ib}(p) \Pi_{jd}(p) - \frac{1}{4} g_{ij}(p) g^{ac}(p) g^{bd}(p) \Pi_{ab}(p) \Pi_{cd}(p) \\ &\quad + m^2 \Theta_i(p) \Theta_j(p) - \frac{1}{2} m^2 g_{ij}(p) g^{ab}(p) \Theta_a(p) \Theta_b(p), \end{aligned}$$

where Π is defined in eq. (3.2.24).

With the choice of a quasi-free Hadamard state τ we can promote $T_{ij}^{\mathcal{M}}$ to the renormalized quantum stress-energy tensor $\mathcal{T}_{\tau ij}^{\mathcal{M}}$ simply via point-splitting (refer to [Wal94, eq. 4.6.5, p. 88]): For each $\eta, \xi \in \mathcal{V}_{\tau}^{\mathcal{M}}$, we choose two “near” points p and q in M and a curve γ connecting them and, parallel transporting along the curve γ , we write

$$\begin{aligned} \langle \eta, \mathcal{T}_{\tau}^{\mathcal{M} ij}(p, q) \xi \rangle_{\tau}^{\mathcal{M}} &= g_{bf}(p) Y_{\gamma b}^f \langle \eta, \Pi_{\tau}^{\mathcal{M} ib}(p) \Pi_{\tau}^{\mathcal{M} jd}(q) \xi \rangle_{\tau}^{\mathcal{M}} \\ &\quad - \frac{1}{4} g^{ik}(p) Y_{\gamma k}^j g_{ae}(p) Y_{\gamma c}^e g_{bf}(p) Y_{\gamma d}^f \langle \eta, \Pi_{\tau}^{\mathcal{M} ab}(p) \Pi_{\tau}^{\mathcal{M} cd}(q) \xi \rangle_{\tau}^{\mathcal{M}} \\ &\quad - \frac{1}{2} m^2 g^{ik}(p) Y_{\gamma k}^j g_{ad}(p) Y_{\gamma b}^d \langle \eta, \Psi_{\tau}^{\mathcal{M} a}(p) \Psi_{\tau}^{\mathcal{M} b}(q) \xi \rangle_{\tau}^{\mathcal{M}} \\ &\quad + m^2 \langle \eta, \Psi_{\tau}^{\mathcal{M} i}(p) \Psi_{\tau}^{\mathcal{M} j}(q) \xi \rangle_{\tau}^{\mathcal{M}}, \quad (3.2.30) \end{aligned}$$

where we introduced

$$\Pi_{\tau}^{\mathcal{M}ij}(p) = \nabla^i \Psi_{\tau}^{\mathcal{M}j}(p) - \nabla^j \Psi_{\tau}^{\mathcal{M}i}(p)$$

to shorten the last formula.

All the remarks made for the Klein-Gordon quantized stress-energy tensor hold also in this case. In particular the expression does not depend upon the choice of the curve γ provided that p and q are in a sufficiently small neighborhood so that there exists a unique geodesic connecting them and we choose γ to be such geodesic. Indeed such choice can be done since our scope is to take the coincidence limit $q \rightarrow p$ along γ (once that we are sure that no divergence may arise). Indeed this is not the standard regularization procedure with respect to τ as reference state, but the result differs only by a multiple of the identity operator. Since we are interested in the commutator of the stress-energy tensor with some other operator, for our aims the point-splitting is equivalent to the standard regularization procedure.

In our upcoming theorem the stress-energy tensor will appear only in a commutator with some represented Weyl generator $V_{\tau}^{\mathcal{M}}(\Theta)$, $\Theta \in V$, where $(V, \sigma) = \mathcal{B}(\mathcal{M}, \Lambda^1 M, A)$ for a fixed object $(\mathcal{M}, \Lambda^1 M, A)$ in \mathbf{ghs}^P . From eq. (3.2.30) we realize that it would be useful to evaluate the matrix elements of the commutator of two unsmeared fields and of $\Pi_{\tau}^{\mathcal{M}}(p) \Pi_{\tau}^{\mathcal{M}}(q)$ with an arbitrary represented Weyl generator: we recall eq. (3.2.2) and we evaluate its LHS and its RHS fixing $\eta, \xi \in \mathcal{V}_{\tau}^{\mathcal{M}}, \theta, \theta' \in \Omega_0^1 M$ and $\Theta \in V$ and exploiting eq. (3.2.29) twice (all the equations are written using the abstract index notation):

$$\begin{aligned} \iint_M \langle \eta, [\Psi_{\tau}^{\mathcal{M}i}(p) \Psi_{\tau}^{\mathcal{M}j}(q), V_{\tau}^{\mathcal{M}}(\Theta)] \xi \rangle_{\tau}^{\mathcal{M}} \theta_i(p) \theta'_j(q) d\mu_g(p) d\mu_g(q) \\ = \langle \eta, [\Psi_{\tau}^{\mathcal{M}}(\theta) \Psi_{\tau}^{\mathcal{M}}(\theta'), V_{\tau}^{\mathcal{M}}(\Theta)] \xi \rangle_{\tau}^{\mathcal{M}}. \end{aligned}$$

Now we exploit also the definition of the symplectic form σ (cfr. Proposition 2.3.7):

$$\begin{aligned} -\sigma(f_A \theta, \Theta) \langle \eta, V_{\tau}^{\mathcal{M}}(\Theta) \Psi_{\tau}^{\mathcal{M}}(\theta') \xi \rangle_{\tau}^{\mathcal{M}} \\ = \iint_M \Theta_k(p) g^{ki}(p) \theta_i(p) \theta'_j(q) \langle \eta, V_{\tau}^{\mathcal{M}}(\Theta) \Psi_{\tau}^{\mathcal{M}j}(q) \xi \rangle_{\tau}^{\mathcal{M}} d\mu_g(p) d\mu_g(q), \\ -\sigma(f_A \theta', \Theta) \langle \eta, \Psi_{\tau}^{\mathcal{M}}(\theta) V_{\tau}^{\mathcal{M}}(\Theta) \xi \rangle_{\tau}^{\mathcal{M}} \\ = \iint_M \Theta_k(q) g^{kj}(q) \theta'_j(q) \theta_i(p) \langle \eta, \Psi_{\tau}^{\mathcal{M}i}(p) V_{\tau}^{\mathcal{M}}(\Theta) \xi \rangle_{\tau}^{\mathcal{M}} d\mu_g(p) d\mu_g(q). \end{aligned}$$

From eq. (3.2.2) and the freedom in the choice of θ and θ' we deduce that

$$\begin{aligned} & \langle \eta, [\Psi_{\tau}^{\mathcal{M}i}(p) \Psi_{\tau}^{\mathcal{M}j}(q), V_{\tau}^{\mathcal{M}}(\Theta)] \xi \rangle_{\tau}^{\mathcal{M}} \\ &= \Theta^i(p) \langle \eta, V_{\tau}^{\mathcal{M}}(\Theta) \Psi_{\tau}^{\mathcal{M}j}(q) \xi \rangle_{\tau}^{\mathcal{M}} + \Theta^j(q) \langle \eta, \Psi_{\tau}^{\mathcal{M}i}(p) V_{\tau}^{\mathcal{M}}(\Theta) \xi \rangle_{\tau}^{\mathcal{M}} \end{aligned} \quad (3.2.31)$$

for each $\Theta \in V$ and each $\eta, \xi \in \mathcal{V}_{\tau}^{\mathcal{M}}$. From the last equation we deduce also that

$$\begin{aligned} & \langle \eta, [\Pi_{\tau}^{\mathcal{M}ij}(p) \Pi_{\tau}^{\mathcal{M}kl}(q), V_{\tau}^{\mathcal{M}}(\Theta)] \xi \rangle_{\tau}^{\mathcal{M}} \\ &= \Pi^{ij}(p) \langle \eta, V_{\tau}^{\mathcal{M}}(\Theta) \Pi_{\tau}^{\mathcal{M}kl}(q) \xi \rangle_{\tau}^{\mathcal{M}} + \Pi^{kl}(q) \langle \eta, \Pi_{\tau}^{\mathcal{M},ij}(p) V_{\tau}^{\mathcal{M}}(\Theta) \xi \rangle_{\tau}^{\mathcal{M}}. \end{aligned} \quad (3.2.32)$$

Main theorem

We devoted the discussion from the beginning of the current subsection to prepare all the material needed to state and prove a theorem about the compatibility between the action of the functional derivative of the relative Cauchy evolution with respect of the spacetime metric and the stress-energy tensor, namely a result similar to the one found in Subsection 3.2.2 for the Klein-Gordon field.

Theorem 3.2.6. *Let $\mathcal{A} : \mathbf{ghs}^P \rightarrow \mathbf{alg}$ be the locally covariant quantum field theory for the Proca field built in Subsection 2.3.2 and let $(\mathcal{M}, \Lambda^1 M, A)$ be an object of the category \mathbf{ghs}^P (see Definition 2.3.8). Consider a quasi-free Hadamard state τ on the CCR representation $(\mathcal{V}, V) = \mathcal{A}(\mathcal{M}, \Lambda^1 M, A)$ and denote the GNS triple induced by τ with $(\mathcal{H}_{\tau}^{\mathcal{M}}, \pi_{\tau}^{\mathcal{M}}, \Omega_{\tau}^{\mathcal{M}})$. We denote with $V_{\tau}^{\mathcal{M}}$ the represented counterpart of the Weyl map V (cfr. eq. (3.2.26)) and with $\mathcal{T}_{\tau}^{\mathcal{M}}$ the quantum stress-energy tensor for the Proca field on \mathcal{M} obtained using the point-splitting procedure in the representation induced by the state τ (cfr. eq. (3.2.30)). Then there exists a dense subspace $\mathcal{V}_{\tau}^{\mathcal{M}}$ of $\mathcal{H}_{\tau}^{\mathcal{M}}$ such that*

$$\frac{\delta}{\delta h} \pi_{\tau}^{\mathcal{M}}(R_h^{\mathcal{M}}(V(\Theta))) = -\frac{i}{2} [\mathcal{T}_{\tau}^{\mathcal{M}}, V_{\tau}^{\mathcal{M}}(\Theta)] \quad \forall \Theta \in V$$

in the sense of quadratic forms on $\mathcal{V}_{\tau}^{\mathcal{M}}$.

Proof. We consider the dense subspace $\mathcal{V}_{\tau}^{\mathcal{M}}$ of $\mathcal{H}_{\tau}^{\mathcal{M}}$ whose existence is assured by the choice of a quasi-free Hadamard state τ on the CCR representation (\mathcal{V}, V) (see few lines before eq. (3.2.28)).

We fix $\xi \in \mathcal{V}_{\tau}^{\mathcal{M}}$, $\Theta \in V$, a compact subset K of M and 1-parameter family of globally hyperbolic perturbations $(-1, 1) \rightarrow GHP(\mathcal{M}, K)$, $s \mapsto h^s$ such that $h^0 = 0$ and we adopt the notation $\delta_s = d/ds|_0$. We reformulate the thesis in a convenient manner imitating the first part of the proof of the similar theorem for the Klein-Gordon field, namely Theorem 3.2.4. The only difference is that here we consider the results for the Proca field in place of the similar results for the Klein-Gordon field. To be precise, we use eq. (3.2.22) in place of eq. (3.2.4), eq. (3.2.28) in place

of eq. (3.2.12), eq. (3.2.23) and eq. (3.2.27) respectively in place of eq. (3.2.7) and eq. (3.2.11). In this way we obtain the following equivalent formulation of our thesis:

$$\underbrace{\langle \xi, \{ \Psi_\tau^\mathcal{M}(\delta_s A[h^s] \Theta), V_\tau^\mathcal{M}(\Theta) \} \xi \rangle_\tau^\mathcal{M}}_{\text{L}} = - \underbrace{\int_M (\delta_s h^s) \left(\langle \xi, [\mathcal{T}_\tau^\mathcal{M}, V_\tau^\mathcal{M}(\Theta)] \xi \rangle_\tau^\mathcal{M} \right)}_{\text{R}} d\mu_g,$$

where the dual pairing between $T^*M \otimes_s T^*M$ and $TM \otimes_s TM$ is considered in the integrand appearing on the RHS.

We begin with the analysis of the LHS of the last equation (denoted by L). The RHS (denoted by R) will be discussed later. We exploit the relation between smeared and unsmeared fields, eq. (3.2.29):

$$\text{L} = \int_M ((\delta_s A[h^s] \Theta)(p)) \left(\langle \xi, \{ \Psi_\tau^\mathcal{M}(p), V_\tau^\mathcal{M}(\Theta) \} \xi \rangle_\tau^\mathcal{M} \right) d\mu_g,$$

where the dual pairing between T^*M and TM is considered. In order to find an expression for L in local coordinates, we repeat the construction performed in the proof of Theorem 3.2.4 to obtain a convenient family of oriented coordinate neighborhoods. In this way we find a finite family $\{(U_\alpha, V_\alpha, \phi_\alpha)\}$ obtained choosing from a locally finite covering of M constituted by oriented coordinate neighborhoods all the elements that intersect the fixed compact subset K of M (which includes the support of the coefficients appearing in $\delta_s A[h^s]$). We stress that the choice of the oriented coordinate neighborhoods is made in such a way that $|\det g| = 1$. At the same time we consider only the corresponding members $\{\chi_\alpha\}$ in the partition of unity subordinate to the original locally finite covering. Using this finite collection of coordinate neighborhoods, together with the corresponding members of the partition of unity, we can obtain the expression of L in local coordinates:

$$\text{L} = \sum_\alpha \int_{V_\alpha} \chi_\alpha \langle \xi, \{ \Psi_\tau^{\mathcal{M}^i}(x), V_\tau^\mathcal{M}(\Theta) \} \xi \rangle_\tau^\mathcal{M} (\delta_s A[h^s] \Theta)_i(x) dV,$$

where dV denotes the standard volume form on \mathbb{R}^4 and all the sections that appear inside the integral are now written in local coordinates⁵. It is convenient to define the section

$$\begin{aligned} \zeta : M &\rightarrow T_{\mathbb{C}}M \\ p &\mapsto \langle \xi, \{ \Psi_\tau^\mathcal{M}(p), V_\tau^\mathcal{M}(\Theta) \} \xi \rangle_\tau^\mathcal{M}, \end{aligned}$$

⁵by this we mean that, inside the integral over V_α , $\delta_s A[h^s] \Theta$ now denotes the push-forward through ϕ_α of the original $\delta_s A[h^s] \Theta$ restricted to U_α and similarly for the other sections inside the integral

where $T_{\mathbb{C}}M$ is the complex vector bundle obtained taking the tensor product of each fiber in TM with \mathbb{C} . Now we use eq. (3.2.25). In this way we obtain

$$\begin{aligned} \mathbf{L} = & \underbrace{\sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^k (\nabla^i \Pi_k^j) \delta_s h_{ij}^s dV}_{\mathbf{L}_1} \\ & + \underbrace{\sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^k \Pi_{lk} \delta_s \Gamma [h^s]_{ij}^l g^{ij} dV}_{\mathbf{L}_2} + \underbrace{\sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^k \Pi_{jl} g^{ij} \delta_s \Gamma [h^s]_{ik}^l dV}_{\mathbf{L}_3}, \end{aligned}$$

where Π is defined as in eq. (3.2.24) and the dependence of the integrand on the point $x \in V_{\alpha}$ is now understood. We denote the first addend appearing on the RHS of the last equation with \mathbf{L}_1 and the others with \mathbf{L}_2 and \mathbf{L}_3 . We integrate \mathbf{L}_1 by parts noting that χ_{α} is null on the boundary of V_{α} , hence no surface term appears:

$$\begin{aligned} \mathbf{L}_1 = & \underbrace{- \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} (\nabla^i \zeta^k) \Pi_k^j \delta_s h_{ij}^s dV}_{\mathbf{X}} \\ & - \underbrace{\sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta_k \Pi^{jk} \nabla^i \delta_s h_{ij}^s dV}_{\mathbf{L}_4} - \underbrace{\sum_{\alpha} \int_{V_{\alpha}} (\nabla^i \chi_{\alpha}) \zeta^k \Pi_k^j \delta_s h_{ij}^s dV}_{=0}. \end{aligned}$$

The last term in the equation above vanishes because the sum of χ_{α} gives 1 on each point of the support of $\delta_s h^s$, hence the sum of their derivatives is null on such region. We denote the first of the remaining terms with \mathbf{X} and the second with \mathbf{L}_4 . At the present moment we have

$$\mathbf{L} = \mathbf{X} + \mathbf{L}_2 + \mathbf{L}_3 + \mathbf{L}_4.$$

Now we investigate \mathbf{R} expressing it in the local coordinates $\{(U_{\alpha}, V_{\alpha}, \phi_{\alpha})\}$:

$$\mathbf{R} = - \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \langle \xi, [\mathcal{T}_{\tau}^{\mathcal{M}ij}(x), V_{\tau}^{\mathcal{M}}(\Theta)] \xi \rangle_{\tau}^{\mathcal{M}} \delta_s h_{ij}^s dV.$$

Now recall the expression of the quantized stress-energy tensor, eq. (3.2.30), and the commutation relation found in eq. (3.2.31) and in eq. (3.2.32) and use these data to evaluate $\langle \xi, [\mathcal{T}_{\tau}^{\mathcal{M}ij}(p, q), V_{\tau}^{\mathcal{M}}(\Theta)] \xi \rangle_{\tau}^{\mathcal{M}}$. That done, observe that no divergence arises in the limit $q \rightarrow p$. Hence we can take the coincidence limit as required by the point-splitting procedure and insert the result in the last equation. Exploiting the symmetry of $\delta_s h^s$ and g , we manage to simplify the result (matrix elements of anticommutators should appear). We obtain the following expression (as above we

replace $\langle \xi, \{ \Psi_{\tau}^{\mathcal{M}}(x), V_{\tau}^{\mathcal{M}}(\Theta) \} \xi \rangle_{\tau}^{\mathcal{M}}$ with ζ):

$$\begin{aligned} R = & \overbrace{- \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} (\nabla^i \zeta^b - \nabla^b \zeta^i) \Pi^j_b \delta_s h_{ij}^s dV - m^2 \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^i \Theta^j \delta_s h_{ij}^s dV}^{R_1} \\ & + \underbrace{\frac{1}{4} \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \Pi_{ab} (\nabla^a \zeta^b - \nabla^b \zeta^a) \delta_s h_{ij}^s g^{ij} dV + \frac{1}{2} m^2 \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^a \Theta_a \delta_s h_{ij}^s g^{ij} dV}_{R_2}. \end{aligned}$$

We denote the term on the first line of the RHS in the last equation with R_1 and that on the second line with R_2 . In first place we evaluate R_1 performing a partial integration on its first term (we directly omit the term containing derivatives of the functions χ_{α} its contribution being null):

$$\begin{aligned} R_1 = & \overbrace{- \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} (\nabla^i \zeta^k) \Pi^j_k \delta_s h_{ij}^s dV}^{=X} \\ & + \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} (\nabla^b \zeta^i) \Pi^j_b \delta_s h_{ij}^s dV - m^2 \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^i \Theta^j \delta_s h_{ij}^s dV \\ = & X - \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^i \Pi^{jb} \nabla_b \delta_s h_{ij}^s dV + \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^i g^{jk} \underbrace{(\nabla^b \Pi_{bk} - m^2 \Theta_k)}_{=0} \delta_s h_{ij}^s dV. \end{aligned}$$

It appears the term X already found in L and with trivial manipulations on summation indices we are able to show a term involving the LHS of the Proca equation (cfr. eq. (2.3.5) bearing in mind the definition of Π given in eq. (3.2.24)). In this way we get rid of another term since $\Theta \in V$, hence it is a solution of the Proca equation. At the moment we have

$$R_1 = X - \underbrace{\sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^k \Pi^{jb} \nabla_b \delta_s h_{kj}^s dV}_{R_3} = X + R_3.$$

In second place we evaluate R_2 proceeding with the same approach. First of all we notice that we can exploit the antisymmetry of Π to simplify a little bit the first integral. Then we partially integrate such term with the intention of finding another integrand that explicitly exhibits the structure of the Proca equation so that we can get rid of it too (again we omit at all the null term containing derivatives of χ_{α}):

$$\begin{aligned} 2R_2 = & \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \Pi_{ab} (\nabla^a \zeta^b) \delta_s h_{ij}^s g^{ij} dV + m^2 \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^a \Theta_a \delta_s h_{ij}^s g^{ij} dV \\ = & - \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^b \Pi_{ab} \nabla^a \delta_s h_{ij}^s g^{ij} dV - \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^b \underbrace{(\nabla^a \Pi_{ab} - m^2 \Theta_b)}_{=0} \delta_s h_{ij}^s g^{ij} dV. \end{aligned}$$

Therefore, renaming some summation indices, we obtain the following result:

$$R_2 = -\frac{1}{2} \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^k \Pi_{lk} \nabla^l \delta_s h_{ij}^s g^{ij} dV.$$

At this stage our thesis $L = R$ is reduced to the following identity:

$$L_2 + L_3 + L_4 = R_2 + R_3. \quad (3.2.33)$$

The remaining part of this proof is similar to the end of the proof of Theorem 3.2.4. To be precise, this time we will prove two identities that hold everywhere on M :

$$\Pi_{jl} g^{ij} \delta_s \Gamma [h^s]_{ik}^l = -\Pi^{jb} \nabla_b \delta_s h_{kj}^s, \quad (3.2.34)$$

$$\delta_s \Gamma [h^s]_{ij}^l g^{ij} - g^{lj} \nabla^i \delta_s h_{ij}^s = -\frac{1}{2} \nabla^l \delta_s h_{ij}^s g^{ij}. \quad (3.2.35)$$

A cursory glance to such identities shows that the first one entails $L_3 = R_3$ (it is sufficient to contract it with $\chi_{\alpha} \zeta^k$ on each V_{α} and then integrate over V_{α} and take the sum over α), while the second entails $L_2 + L_4 = R_2$ (now you should contract with $\chi_{\alpha} \zeta^k \Pi_{lk}$ and then proceed as in the other case). Hence these identities together imply our thesis, eq. (3.2.33). We prove them fixing a point p of M and choosing Riemannian normal coordinates in a (sufficiently small) neighborhood of p (cfr. e.g. [Wal84, Sect. 3.3, p. 42]) so that the Christoffel symbols of the connection ∇ are null at p (note that nothing can be said about the Christoffel symbols of a “perturbed” connection $\nabla [h^s]$).

We begin evaluating the LHS of the first identity, eq. (3.2.34), with the help of eq. (3.2.20):

$$\begin{aligned} \Pi_{jl} g^{ij} \delta_s \Gamma [h^s]_{ik}^l &= \Pi^{im} \frac{1}{2} (\partial_i \delta_s h_{mk}^s + \partial_k \delta_s h_{im}^s - \partial_m \delta_s h_{ik}^s) \\ &= \Pi^{im} \partial_i \delta_s h_{mk}^s \\ &= -\Pi^{jb} \nabla_b \delta_s h_{kj}^s, \end{aligned}$$

where we exploited the symmetry of $\delta_s h^s$, the antisymmetry of Π (note that in particular $\Pi^{im} \partial_k \delta_s h_{im}^s = 0$) and we renamed some summation indices for convenience. This calculation shows that eq. (3.2.34) actually holds.

Now we focus on the second identity, eq. (3.2.35). Specifically we evaluate the first term on its LHS using eq. (3.2.21), exploiting the symmetry of $\delta_s h_{ij}^s$, renaming some summation indices and bearing in mind that our choice of coordinates allows

us to replace ∇ with ∂ and vice versa at the fixed point p :

$$\begin{aligned}\delta_s \Gamma [h^s]_{ij}^l g^{ij} &= g^{ij} g^{lk} \partial_i \delta_s h_{kj}^s - \frac{1}{2} g^{ij} g^{lk} \partial_k \delta_s h_{ij}^s \\ &= g^{lj} \nabla^i \delta_s h_{ij}^s - \frac{1}{2} \nabla^l \delta_s h_{ij}^s g^{ij}.\end{aligned}$$

This shows that eq. (3.2.35) holds too, hence the proof is complete. \square

3.2.4 Relative Cauchy evolution for the electromagnetic field

The last question we try to answer deals with the agreement between the action of the functional derivative of the relative Cauchy evolution for the electromagnetic field and its quantized stress-energy tensor. To tackle such problem we resort to our discussion about the locally covariant quantum field theory for the electromagnetic field (cfr. Subsection 2.3.3).

Our approach will be similar to the last two subsections, but now we consider the electromagnetic field, hence we adopt the notation introduced in Subsection 2.3.3 and we refer to the results proved there. In particular here we consider the category \mathbf{ghs}^{EM} (see Definition 2.3.15) and the covariant functor $\mathcal{B} : \mathbf{ghs}^{EM} \rightarrow \mathbf{ssp}$ describing the classical theory of the electromagnetic field which fulfils both the causality condition and the time slice axiom in the sense of functors describing classical field theories (see Theorem 2.3.17). Having \mathcal{B} at disposal, we follow the usual procedure (see Subsection 2.2.2) to obtain the locally covariant quantum field theory $\mathcal{A} : \mathbf{ghs}^{EM} \rightarrow \mathbf{alg}$ for the electromagnetic field, i.e. we take the composition of \mathcal{B} with the quantization functor $\mathcal{C} : \mathbf{ssp} \rightarrow \mathbf{alg}$. By virtue of the properties enjoyed by \mathcal{B} , we deduce that \mathcal{A} is causal and fulfils the time slice axiom in the sense of LCQFTs (for the details refer to Subsection 2.3.3). In particular, the fulfilment of the time slice axiom is essential for the upcoming discussion.

Relative Cauchy evolution for the classical electromagnetic field

The first building block for our final theorem is an expression for the relative Cauchy evolution for the electromagnetic field at a classical level. Such result will be achieved with the help of the next proposition. We remind the reader that an object of \mathbf{ghs}^{EM} is a triple $(\mathcal{M}, \Lambda^1 M, A)$ where $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ is a globally hyperbolic spacetime, $\Lambda^1 M$ denotes the cotangent bundle over M which we endow with the inner product $\langle \cdot, \cdot \rangle_{g,1}$ induced by the metric g and A is the linear differential operator δd acting on sections in $\Lambda^1 M$ (note that such operator depends on the metric g through the codifferential δ).

As we did for the Klein-Gordon field and the Proca field, we are going to take into account an object of \mathbf{ghs}^{EM} denoted by $(\mathcal{M}|_O, \Lambda^1 O, A|_O)$ for some \mathcal{M} -causally

convex connected open subset O of M . To see how such object is defined and realize that it is actually an object of \mathbf{ghs}^{EM} refer to the first part of the proof of Proposition 3.2.3 replacing Λ^0 with Λ^1 .

A slight difference appears at the level of morphisms since now we consider only push-forwards of morphisms of \mathbf{ghs} . This has to be intended in a proper sense, namely that of Remark 2.3.11: We call push-forward of a morphism $\psi \in \mathbf{Mor}_{\mathbf{ghs}}(\mathcal{M}, \mathcal{N})$ the composition of the inclusion map of the proper tensor bundle over $\psi(M)$ into the tensor bundle over N of the same type and the push-forward $\psi'_* = (\psi'^{-1})^* : \Lambda M \rightarrow \Lambda \psi(M)$ through the isometric diffeomorphism $\psi' : M \rightarrow \psi(M)$ induced by ψ . For example a morphism (ψ, ψ_*) of \mathbf{ghs}^{EM} from $(\mathcal{M}, \Lambda^1 M, A)$ to $(\mathcal{N}, \Lambda^1 N, B)$ acts on a element of $\Lambda^k M$ as $\iota_{\Lambda^k M}^{\Lambda^k N} \circ (\psi'^{-1})^*$.

Here we are interested in morphisms of \mathbf{ghs} that are generated by the inclusion maps of a causally convex connected open subset of a globally hyperbolic spacetime into the whole spacetime. In such cases the induced isometric diffeomorphism is nothing but the identity map of the subset, hence the push-forward (in the sense specified above) reduces to the inclusion map between the proper tensor bundles.

Proposition 3.2.7. *Let $(\mathcal{M}, \Lambda^1 M, A)$ be an object of \mathbf{ghs}^{EM} and let O be an \mathcal{M} -causally convex connected open subset of M including a smooth spacelike Cauchy surface Σ for \mathcal{M} . Consider the object $(\mathcal{M}|_O, \Lambda^1 O, A|_O)$ of \mathbf{ghs}^{EM} and the morphism $(\iota_O^M, \iota_{O*}^M)$ of \mathbf{ghs}^{EM} from $(\mathcal{M}|_O, \Lambda^1 O, A|_O)$ to $(\mathcal{M}, \Lambda^1 M, A)$ induced by the inclusion map $\iota_O^M : O \rightarrow M$. Then there exists a partition of unity $\{\chi^a, \chi^r\}$ on M such that the inverse $\mathcal{B}(\iota_O^M, \iota_{O*}^M)^{-1}$ of the bijective morphism $\mathcal{B}(\iota_O^M, \iota_{O*}^M)$ of \mathbf{ssp} from $(V, \sigma) = \mathcal{B}(\mathcal{M}|_O, \Lambda^1 O, A|_O)$ to $(W, \omega) = \mathcal{B}(\mathcal{M}, \Lambda^1 M, A)$ satisfies the following equation:*

$$\mathcal{B}(\iota_O^M, \iota_{O*}^M)^{-1} [\mathbf{A}]_M = \left[\pm e_{A|_O} \left(\text{res}_{\iota_{O*}^M} (A(\chi^{a/r} \mathbf{A})) \right) \right]_O \quad \forall [\mathbf{A}]_M \in W,$$

where \mathbf{A} is a representative of the equivalence class $[\mathbf{A}]_M$, $e_{A|_O}$ is the causal propagator for the formally selfadjoint normally hyperbolic operator $\square_1|_O = (A + d\delta)|_O$ and the restriction map is defined in Lemma 2.2.4.

Proof. We apply the procedure presented in the first part of the proof of Proposition 3.2.3 to choose two smooth spacelike Cauchy surfaces $\Sigma_{-\varepsilon}$ and Σ_ε for \mathcal{M} contained in O among the smooth spacelike Cauchy surfaces in the foliation of \mathcal{M} induced by Σ . With this choice we consider the open covering $\{I_+^\mathcal{M}(\Sigma_{-\varepsilon}), I_-^\mathcal{M}(\Sigma_\varepsilon)\}$ of M and its subordinate partition of unity $\{\chi^a, \chi^r\}$.

Take now $[\mathbf{A}]_M \in W$. As a consequence of the construction of the functor \mathcal{B} , $W = \{[e_A \theta]_M : \theta \in \Omega_{0,\delta}^1 M\}$ (we remind the reader that $\Omega_{0,\delta}^1 M$ denotes the space of compactly supported coclosed 1-forms). Hence, choosing a representative \mathbf{A} of the class $[\mathbf{A}]_M$, we also find $\theta \in \Omega_{0,\delta}^1 M$ and therefore we deduce $\text{supp}(\mathbf{A}) \subseteq J^\mathcal{M}(K)$ for $K = \text{supp}(\theta)$, which is a compact subset of M . If we define $\mathbf{A}^{a/r} = \chi^{a/r} \mathbf{A}$, we see

that

$$\text{supp}(\mathbf{A}^{a/r}) \subseteq J_{\pm}^{\mathcal{M}}(\Sigma_{\mp\varepsilon}).$$

From the last inclusion it follows that $\mathbf{A}^{a/r}$ is an element of $\Omega^1 M$ with \mathcal{M} -past/future compact support. Moreover we know that $\square_1 \mathbf{A} = 0$ and $\delta \mathbf{A} = 0$ because $\mathbf{A} = e_A \theta$, $\delta \theta = 0$ and $\delta e_A \theta = e_A \delta \theta$ (see Lemma 2.3.5). This entails that $A\mathbf{A} = 0$. From this fact, together with $\chi^a + \chi^r = 1$, we deduce

$$\begin{aligned} A\mathbf{A}^a &= -A\mathbf{A}^r, \\ \delta \mathbf{A}^a &= -\delta \mathbf{A}^r, \end{aligned}$$

hence

$$\begin{aligned} \text{supp}(A\mathbf{A}^a) &\subseteq J^{\mathcal{M}}(K) \cap J_+^{\mathcal{M}}(\Sigma_{-\varepsilon}) \cap J_-^{\mathcal{M}}(\Sigma_{\varepsilon}) \subseteq O, \\ \text{supp}(\delta \mathbf{A}^a) &\subseteq J^{\mathcal{M}}(K) \cap J_+^{\mathcal{M}}(\Sigma_{-\varepsilon}) \cap J_-^{\mathcal{M}}(\Sigma_{\varepsilon}) \subseteq O. \end{aligned}$$

Exploiting Proposition 1.2.18, we realize that both $A\mathbf{A}^{a/r}$ and $\delta \mathbf{A}^{a/r}$ fall in $\Omega_0^1 M$ and their supports are contained in O . At this point we know that we can apply the restriction map⁶ to $A\mathbf{A}^{a/r}$ (and indeed also to $\delta \mathbf{A}^{a/r}$) in order to obtain an element of $\Omega_0^1 O$:

$$\theta' = \text{res}_{\iota_{O*}^M}(A(\chi^{a/r} \mathbf{A})) \in \Omega_0^1 O.$$

One can almost immediately recognize that $\delta \theta' = 0$ because $\delta \circ A = \delta \circ \delta \circ d = 0$ and

$$\delta \circ \text{res}_{\psi_*} = \text{res}_{\psi_*} \circ \delta$$

for each morphism (ψ, ψ_*) of \mathbf{ghs}^{EM} because the push-forward intertwines with both d and δ (see the footnote at page 141). This proves that it makes sense to consider

$$\left[\pm e_{A|_O} \left(\text{res}_{\iota_{O*}^M}(A(\chi^{a/r} \mathbf{A})) \right) \right]_O \in V = \{ [e_{A|_O} \theta']_M : \theta' \in \Omega_{0,\delta}^1 O \}.$$

Suppose that we choose a different representative \mathbf{A}' of $[\mathbf{A}]_M$. We obtain

$$\left[\pm e_{A|_O} \left(\text{res}_{\iota_{O*}^M}(A(\chi^{a/r} \mathbf{A}')) \right) \right]_O \in V.$$

Indeed we know that $d(\mathbf{A} - \mathbf{A}') = 0$ because \mathbf{A} and \mathbf{A}' are in the same equivalence class and we wonder if the new element of V coincides with the old one. To answer such question we have to take a representative from each of the elements of V considered and show that they differ by a closed 1-form. Since all operators involved are linear,

⁶the definition of the restriction map in the general context of vector bundles was given in Lemma 2.2.4.

it is sufficient to show that

$$d \left(e_{A|_O} \left(\text{res}_{\iota_{O*}^M} \left(A \left(\chi^{a/r} \tilde{\mathbf{A}} \right) \right) \right) \right) = 0,$$

where $\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{A}'$. We apply again Lemma 2.3.5 and, bearing in mind the footnote at page 141, we obtain

$$d \left(e_{A|_O} \left(\text{res}_{\iota_{O*}^M} \left(A \left(\chi^{a/r} \tilde{\mathbf{A}} \right) \right) \right) \right) = e_{A|_O} \left(\text{res}_{\iota_{O*}^M} \left(dA \left(\chi^{a/r} \tilde{\mathbf{A}} \right) \right) \right).$$

Since $d\tilde{\mathbf{A}} = 0$, we deduce that $d(\chi^a \tilde{\mathbf{A}}) = -d(\chi^r \tilde{\mathbf{A}})$. From this relation we deduce that $d(\chi^a \tilde{\mathbf{A}})$ has compact support contained in O (the proof is identical to that of the compactness and the inclusion in O of the supports of $A\mathbf{A}^{a/r}$ and $\delta\mathbf{A}^{a/r}$). Moreover $d \circ A = \square_1 \circ d$ and

$$\square_1 \circ \text{res}_{\iota_{O*}^M} = \text{res}_{\iota_{O*}^M} \circ \square_1.$$

From all these observations we conclude that

$$d \left(e_{A|_O} \left(\text{res}_{\iota_{O*}^M} \left(A \left(\chi^{a/r} \tilde{\mathbf{A}} \right) \right) \right) \right) = e_{A|_O} \square_1 \left(\text{res}_{\iota_{O*}^M} \left(d \left(\chi^{a/r} \tilde{\mathbf{A}} \right) \right) \right) = 0.$$

This proves that the map

$$\begin{aligned} \alpha : W &\rightarrow V \\ [\mathbf{A}]_M &\mapsto \left[\pm e_{A|_O} \left(\text{res}_{\iota_{O*}^M} \left(A \left(\chi^{a/r} \mathbf{A} \right) \right) \right) \right]_O, \end{aligned}$$

where \mathbf{A} is a representative of $[\mathbf{A}]_M$, is well defined.

Note that, from the hypothesis made, we know that the image $\iota_O^M(O) = O$ includes a smooth spacelike Cauchy surface for \mathcal{M} . Hence $\mathcal{B}(\iota_O^M, \iota_{O*}^M)^{-1}$ is a morphism of **ssp** from (W, ω) to (V, σ) because the time slice axiom holds for \mathcal{B} (cfr. Theorem 2.3.17). To conclude the proof we must check that $\alpha = \mathcal{B}(\iota_O^M, \iota_{O*}^M)^{-1}$. Take $[\mathbf{A}]_M \in W$ and one of its representatives \mathbf{A} , consider a partition of unity $\{\chi^a, \chi^b\}$ built following the prescriptions given above and define $\mathbf{A}^{a/r} = \chi^{a/r} \mathbf{A}$. Recalling Lemma 2.3.16 and observing that the restriction map followed by the corresponding extension leaves the argument of the restriction unchanged, we find

$$\begin{aligned} \mathcal{B}(\iota_O^M, \iota_{O*}^M)(\alpha[\mathbf{A}]_M) &= \mathcal{B}(\iota_O^M, \iota_{O*}^M) \left[\pm e_{A|_O} \left(\text{res}_{\iota_{O*}^M} \left(A \mathbf{A}^{a/r} \right) \right) \right]_O \\ &= \left[\pm e_A \left(\text{ext}_{\iota_{O*}^M} \circ \text{res}_{\iota_{O*}^M} \right) A \mathbf{A}^{a/r} \right]_M \\ &= \left[\pm e_A A \mathbf{A}^{a/r} \right]_M. \end{aligned}$$

The support properties of $\mathbf{A}^{a/r}$, $A\mathbf{A}^{a/r}$ and $\delta\mathbf{A}^{a/r}$ allow us to apply Lemma 2.3.5 and

Lemma 1.3.17:

$$\begin{aligned}
\pm e_A A A^{a/r} &= \pm (e_A \square_1 A^{a/r} - e_A d \delta A^{a/r}) \\
&= \pm [(e_A^a \square_1 A^{a/r} - e_A^r \square_1 A^{a/r}) - e_A d \delta A^{a/r}] \\
&= A^a + A^r \mp d e_A \delta A^{a/r} \\
&= A \mp d e_A \delta A^{a/r}.
\end{aligned}$$

The result of the last calculation entails that $\pm e_A A A^{a/r}$ and A are gauge equivalent, hence

$$[\pm e_A A A^{a/r}]_M = [A]_M.$$

With this we conclude

$$\mathcal{B}(\iota_O^M, \iota_{O*}^M)(\alpha[A]_M) = [A]_M = \mathcal{B}(\iota_O^M, \iota_{O*}^M)\left(\mathcal{B}(\iota_O^M, \iota_{O*}^M)^{-1}[A]_M\right) \quad \forall [A]_M \in W.$$

Since $\mathcal{B}(\iota_O^M, \iota_{O*}^M)$ is injective, the last equation entails

$$\alpha[A]_M = \mathcal{B}(\iota_O^M, \iota_{O*}^M)^{-1}[A]_M \quad \forall [A]_M \in W,$$

therefore we realize that the thesis actually holds. \square

We specialize the definition of the RCE to the case of the electromagnetic field. Consider an object $(\mathcal{M}, \Lambda^1 M, A)$ of \mathbf{ghs}^{EM} , take $h \in GHP(\mathcal{M})$ and recall the definitions of the morphisms $v_{\pm}^{\mathcal{M}}[h]$ and $j_{\pm}^{\mathcal{M}}[h]$ of the category \mathbf{ghs} introduced before Definition 3.1.3. Together with the perturbed spacetime $\mathcal{M}[h]$, we must also consider the effects of the perturbation h on the vector bundle (specifically on the inner product defined on it) and on the linear differential operator $A = \delta d$. The inner product on the cotangent bundle over the perturbed spacetime is induced by the perturbed metric $g_h = g + h$ and the perturbed linear differential operator is $A[h] = \delta[h]d$, where $\delta[h]$ is the codifferential defined on $\mathcal{M}[h]$. It can be useful to consider also the perturbed d'Alembert operator $\square_1[h] = \delta[h]d + d\delta[h]$ acting on 1-form over $\mathcal{M}[h]$. We take into account the inclusion map $\iota_{M_{\pm}^*}^M$, where $M_{\pm} = M \setminus J_{\mp}^{\mathcal{M}}(\text{supp}(h))$ in accordance with the definitions of $v_{\pm}^{\mathcal{M}}[h]$ and $j_{\pm}^{\mathcal{M}}[h]$. Compatibility of $(\iota_{M_{\pm}^*}^M, \iota_{M_{\pm}^*}^M)$ with both d and δ holds (see the footnote at page 141):

$$\begin{aligned}
\text{ext}_{\iota_{M_{\pm}^*}^M}(d\theta) &= d\left(\text{ext}_{\iota_{M_{\pm}^*}^M}\theta\right) \quad \forall \theta \in \Omega_0^k M_{\pm}, \\
\text{ext}_{\iota_{M_{\pm}^*}^M}(\delta\theta) &= \delta\left(\text{ext}_{\iota_{M_{\pm}^*}^M}\theta\right) \quad \forall \theta \in \Omega_0^k M_{\pm}.
\end{aligned}$$

Since the effects of the perturbation h are relevant only inside $\text{supp}(h)$, we realize that $\delta[h]$ and δ act exactly in the same way on sections supported outside $\text{supp}(h)$. Together with $A|_{M_{\pm}}$, we may consider $A[h]|_{M_{\pm}}$ and we immediately recognize that they are the same linear differential operator acting on sections in $\Lambda^1 M_{\pm}$ (we denote

both of them with $A_{\pm}[h]$). All these observations are made in order to introduce the objects $(\mathcal{M}[h], \Lambda^1 M, A[h])$ and $(\mathcal{M}_{\pm}[h], \Lambda^1 M_{\pm}, A_{\pm}[h])$ of \mathbf{ghs}^{EM} and to interpret $(\iota_{M_{\pm}}^M, \iota_{M_{\pm}*}^M)$ both as a morphism from $(\mathcal{M}_{\pm}[h], \Lambda^1 M_{\pm}, A_{\pm}[h])$ to $(\mathcal{M}, \Lambda^1 M, A)$ and as a morphism from $(\mathcal{M}_{\pm}[h], \Lambda^1 M_{\pm}, A_{\pm}[h])$ to $(\mathcal{M}[h], \Lambda^1 M, A[h])$ (note the analogy with the definitions of $i_{\pm}^{\mathcal{M}}[h]$ and $j_{\pm}^{\mathcal{M}}[h]$ as different morphisms obtained from the inclusion map $\iota_{M_{\pm}}^M$). We denote such morphisms in the following way:

$$\begin{aligned} (i_{\pm}^{\mathcal{M}}[h], i_{\pm*}^{\mathcal{M}}[h]) &\in \mathbf{Mor}_{\mathbf{ghs}^{EM}}((\mathcal{M}_{\pm}[h], \Lambda^1 M_{\pm}, A_{\pm}[h]), (\mathcal{M}, \Lambda^1 M, A)), \\ (j_{\pm}^{\mathcal{M}}[h], j_{\pm*}^{\mathcal{M}}[h]) &\in \mathbf{Mor}_{\mathbf{ghs}^{EM}}((\mathcal{M}_{\pm}[h], \Lambda^1 M_{\pm}, A_{\pm}[h]), (\mathcal{M}[h], \Lambda^1 M, A[h])). \end{aligned}$$

Denote with \mathcal{A} the LCQFT (fulfilling both the causality condition and the time slice axiom) built in Subsection 2.3.3. For $(\mathcal{M}, \Lambda^1 M, A) \in \mathbf{Obj}_{\mathbf{ghs}^{EM}}$ and $h \in GHP(\mathcal{M})$ we define the RCE for the electromagnetic field as:

$$\begin{aligned} R_h^{\mathcal{M}} &= \mathcal{A}(i_-^{\mathcal{M}}[h], i_{-*}^{\mathcal{M}}[h]) \circ \mathcal{A}(j_-^{\mathcal{M}}[h], j_{-*}^{\mathcal{M}}[h])^{-1} \\ &\quad \circ \mathcal{A}(j_+^{\mathcal{M}}[h], j_{+*}^{\mathcal{M}}[h]) \circ \mathcal{A}(i_+^{\mathcal{M}}[h], i_{+*}^{\mathcal{M}}[h])^{-1}. \end{aligned}$$

In a similar way one can consider a classical version of the RCE based on the covariant functor \mathcal{B} describing the classical theory of the electromagnetic field (this is actually possible due to version of the time slice axiom satisfied by \mathcal{B} , cfr. Theorem 2.3.17):

$$\begin{aligned} r_h^{\mathcal{M}} &= \mathcal{B}(i_-^{\mathcal{M}}[h], i_{-*}^{\mathcal{M}}[h]) \circ \mathcal{B}(j_-^{\mathcal{M}}[h], j_{-*}^{\mathcal{M}}[h])^{-1} \\ &\quad \circ \mathcal{B}(j_+^{\mathcal{M}}[h], j_{+*}^{\mathcal{M}}[h]) \circ \mathcal{B}(i_+^{\mathcal{M}}[h], i_{+*}^{\mathcal{M}}[h])^{-1}. \end{aligned}$$

Since the LCQFT \mathcal{A} is obtained via composition of \mathcal{B} with the quantization functor \mathcal{C} presented in Subsection 2.2.2, we realize that⁷

$$R_h^{\mathcal{M}} = \mathcal{C}(r_h^{\mathcal{M}}). \quad (3.2.36)$$

We can determine the action of $r_h^{\mathcal{M}}$ applying Proposition 3.2.7 and Lemma 2.3.16. We find proper partitions of unity $\{\chi_+^a, \chi_+^r\}$ and $\{\chi_-^a, \chi_-^r\}$ on M such that we can express the action of $\mathcal{B}(i_+^{\mathcal{M}}[h], i_{+*}^{\mathcal{M}}[h])^{-1}$ and respectively of $\mathcal{B}(j_-^{\mathcal{M}}[h], j_{-*}^{\mathcal{M}}[h])^{-1}$ according to Proposition 3.2.7. If we take $[\mathbf{A}]_M \in \mathcal{B}(\mathcal{M}, \Lambda^1 M, A)$ and evaluate $r_h^{\mathcal{M}}[\mathbf{A}]_M$, we easily obtain the following result:

$$r_h^{\mathcal{M}}[\mathbf{A}]_M = \left[e_A A[h] \left(\chi_-^{a/r} e_{A[h]} A \left(\chi_+^{a/r} \mathbf{A} \right) \right) \right]_M, \quad (3.2.37)$$

whatever choice of the representative \mathbf{A} of the equivalence class $[\mathbf{A}]_M$ we make. The independence on the choice of the representative follows from the fact that the same

⁷this is a direct consequence of the covariant axioms, which are required to be verified by any covariant functor

property holds for all the morphisms that we composed to find the expression above.

To prove our final theorem we will need the expression of $\frac{d}{ds} r_{h^s}^{\mathcal{M}} [\mathbf{A}]_M|_0$ for an arbitrary smooth 1-parameter family of perturbations of the metric. For convenience in the upcoming calculation we will denote $\frac{d}{ds} (\cdot)|_0$ with δ_s . Fix $[\mathbf{A}]_M \in \mathcal{B}(\mathcal{M}, \Lambda^1 M, A)$, a compact subset K of M and a smooth 1-parameter family of globally hyperbolic perturbations $(-1, 1) \rightarrow GHP(\mathcal{M}, K)$, $s \mapsto h^s$ such that $h^0 = 0$. To evaluate $\delta_s r_{h^s}^{\mathcal{M}} [\mathbf{A}]_M$, we start from eq. (3.2.37) with the choice of the superscript r (indeed the choice of a would produce a similar calculation and the same result). In the present situation apparently we would have to consider different partitions of unity $\{\chi_+^a, \chi_+^r\}$ and $\{\chi_-^a, \chi_-^r\}$ for each of the values assumed by s . Anyway such complication can be avoided making an intelligent choice of the smooth spacelike Cauchy surfaces used to define the partitions of unity: We use always the same foliation of \mathcal{M} (induced by some fixed smooth spacelike Cauchy surface Σ for \mathcal{M}) and take the smooth spacelike Cauchy surfaces inside $M_{\pm} = M \setminus J_{\mp}^{\mathcal{M}}(K)$ instead of choosing, for each value of s , a pair of proper smooth spacelike Cauchy surfaces inside $M_{\pm} = M \setminus J_{\mp}^{\mathcal{M}}(\text{supp}(h^s))$. In this way a single choice of the smooth spacelike Cauchy surfaces is satisfactory for each value of s . Such choice is possible because the supports of all the elements h^s in the family of perturbations are controlled by the compact subset K of M .

We fix $[\mathbf{A}]_M$ and we take one of its representatives \mathbf{A} . For convenience we define

$$\mathbf{X}^s = e_A A[h^s] \left(\chi_-^{a/r} e_{A[h^s]} A \left(\chi_+^{a/r} \mathbf{A} \right) \right)$$

so that eq. (3.2.37) becomes $r_{h^s}^{\mathcal{M}} [\mathbf{A}]_M = [\mathbf{X}^s]_M$ and our problem reduces to the search of a convenient vector potential that is gauge equivalent to $\delta_s \mathbf{X}^s$. We try to reproduce the calculations performed in the case of the Klein-Gordon field. The important thing now is that we can add terms to our representative vector potential without changing the equivalence class in which it falls, provided that such terms are closed and coclosed 1-forms: actually this means that we can take Lorentz solutions (refer to Lemma 2.3.13) that are gauge equivalent to our starting Lorentz solution $\delta_s \mathbf{X}^s$. In first place we apply the Leibniz rule (see the footnote at page 170):

$$\delta_s \mathbf{X}^s = e_A \left((\delta_s A[h^s]) (\chi_-^r e_A A (\chi_+^r \mathbf{A})) + A (\chi_-^r (\delta_s e_{A[h^s]}) A (\chi_+^r \mathbf{A})) \right).$$

We focus on the first addend: On the one hand, following the proof of Proposition 3.2.7 (we are considering $M_- = M \setminus J_+^{\mathcal{M}}(K)$ as O), we can easily see that $\text{supp}(\chi_-^r) \subseteq J_-^{\mathcal{M}}(M_-)$, while on the other hand $\delta_s A[h^s]$ can have non null coefficients only inside K . This entails that

$$(\delta_s A[h^s]) (\chi_-^r e_A A (\chi_+^r \mathbf{A})) = 0,$$

so that we obtain

$$\delta_s \mathbf{X}^s = e_A A \left(\chi_-^r \left(\delta_s e_{A[h^s]} \right) A \left(\chi_+^r \mathbf{A} \right) \right).$$

Recalling again the proof of Proposition 3.2.7, one sees that $A \left(\chi_+^r \mathbf{A} \right) = -A \left(\chi_+^a \mathbf{A} \right)$ and hence deduces that its support is compact and lies in the causal future of a smooth spacelike Cauchy surface for \mathcal{M} included in $M_+ = M \setminus J_-^{\mathcal{M}}(K)$ (that by construction lies outside K and intersects its causal future). On the contrary χ_-^r is supported in the causal past of a smooth spacelike Cauchy surface for \mathcal{M} included in M_- (that by construction lies outside K and intersects its causal past). These observations entail that $\chi_-^r e_{A[h^s]}^a A \left(\chi_+^r \mathbf{A} \right)$ has empty support for each s , hence it is null. Therefore from the last equation we obtain

$$\delta_s \mathbf{X}_s = -e_A A \left(\chi_-^r \left(\delta_s e_{A[h^s]}^r \right) A \left(\chi_+^r \mathbf{A} \right) \right). \quad (3.2.38)$$

Now we take a closer look to the term $e_{A[h^s]}^r A[h^s] \left(\chi_+^r \mathbf{A} \right)$ for an arbitrary but fixed value of s . In order for this term to make sense it must be shown that $A[h^s] \left(\chi_+^r \mathbf{A} \right)$ has compact support. This follows from the the following facts:

- $A \left(\chi_+^r \mathbf{A} \right) = -A \left(\chi_+^a \mathbf{A} \right)$ implies that $A \left(\chi_+^{a/r} \mathbf{A} \right)$ has compact support (note that $\chi_+^{a/r}$ is supported in the causal future/past of a proper smooth spacelike Cauchy surface for \mathcal{M} and remember that $\text{supp}(\mathbf{A}) \subseteq J^{\mathcal{M}}(K')$ for a proper compact subset K' of M);
- $\delta \mathbf{A} = 0$, hence we also have $\delta \left(\chi_+^r \mathbf{A} \right) = -\delta \left(\chi_+^a \mathbf{A} \right)$, which entails that $\delta \left(\chi_+^{a/r} \mathbf{A} \right)$ has compact support by the argument exploited in the previous point;
- $\delta[h^s]$ acts as δ on sections whose support has empty intersection with K , which is compact, hence the support of the codifferential of a section can be enlarged at most by K when we replace δ with the perturbed codifferential $\delta[h^s]$ (obviously the same conclusion holds if we replace A with $A[h^s]$).

These facts imply that

$$\begin{aligned} \text{supp} \left(A[h^s] \left(\chi_+^{a/r} \mathbf{A} \right) \right) &\subseteq \text{supp} \left(A \left(\chi_+^{a/r} \mathbf{A} \right) \right) \cup K, \\ \text{supp} \left(\delta[h^s] \left(\chi_+^{a/r} \mathbf{A} \right) \right) &\subseteq \text{supp} \left(\delta \left(\chi_+^{a/r} \mathbf{A} \right) \right) \cup K, \\ \text{supp} \left(\square_1[h^s] \left(\chi_+^{a/r} \mathbf{A} \right) \right) &\subseteq \text{supp} \left(A \left(\chi_+^{a/r} \mathbf{A} \right) \right) \cup \text{supp} \left(\delta \left(\chi_+^{a/r} \mathbf{A} \right) \right) \cup K, \end{aligned}$$

hence the supports appearing on the LHS of the last inclusions are compact subsets of M since they are closed (by definition of support) and contained in the union of a finite number of compact subsets of M . From the first point above it follows also that $\chi_+^{a/r} \mathbf{A}$ has past/future compact support (we are exploiting Proposition 1.2.18).

Hence we can apply Lemma 1.3.17 and Lemma 2.3.5 to conclude that

$$\begin{aligned} e_{A[h^s]}^{a/r} A[h^s] \left(\chi_+^{a/r} \mathbf{A} \right) &= e_{A[h^s]}^{a/r} \square_1 [h^s] \left(\chi_+^{a/r} \mathbf{A} \right) - e_{A[h^s]}^{a/r} d[h^s] \left(\chi_+^{a/r} \mathbf{A} \right) \\ &= \chi_+^{a/r} \mathbf{A} - d e_{A[h^s]}^{a/r} \delta[h^s] \left(\chi_+^{a/r} \mathbf{A} \right). \end{aligned} \quad (3.2.39)$$

Applying δ_s to both sides of the last equation (with the superscript r) and exploiting the Leibniz rule, we find

$$(\delta_s e_{A[h^s]}^r) A(\chi_+^r \mathbf{A}) + e_A^r (\delta_s A[h^s]) (\chi_+^r \mathbf{A}) = d(\delta_s (e_{A[h^s]}^r \delta[h^s] (\chi_+^r \mathbf{A}))),$$

which can be written as

$$(\delta_s e_{A[h^s]}^r) A(\chi_+^r \mathbf{A}) = -e_A^r (\delta_s A[h^s]) (\chi_+^r \mathbf{A}) + d(\delta_s (e_{A[h^s]}^r \delta[h^s] (\chi_+^r \mathbf{A}))).$$

With this identity we can rewrite eq. (3.2.38):

$$\delta_s \mathbf{X}^s = e_A A(\chi_-^r e_A^r (\delta_s A[h^s]) (\chi_+^r \mathbf{A})) - e_A A(\chi_-^r d(\delta_s (e_{A[h^s]}^r \delta[h^s] (\chi_+^r \mathbf{A}))))).$$

Now we show that the second term appearing on the RHS is both closed and coclosed applying Lemma 2.3.5:

$$\begin{aligned} d e_A A(\chi_-^r d(\delta_s (e_{A[h^s]}^r \delta[h^s] (\chi_+^r \mathbf{A})))) &= e_A \square_2 d(\chi_-^r d(\delta_s (e_{A[h^s]}^r \delta[h^s] (\chi_+^r \mathbf{A})))) = 0, \\ \delta e_A A(\chi_-^r d(\delta_s (e_{A[h^s]}^r \delta[h^s] (\chi_+^r \mathbf{A})))) &= e_A \delta A(\chi_-^r d(\delta_s (e_{A[h^s]}^r \delta[h^s] (\chi_+^r \mathbf{A})))) = 0. \end{aligned}$$

Therefore we are allowed to replace the representative $\delta_s \mathbf{X}^s$ with the representative

$$\mathbf{Y} = e_A A(\chi_-^r e_A^r (\delta_s A[h^s]) (\chi_+^r \mathbf{A}))$$

without changing the equivalence class.

Observing that χ_+^a is supported inside $J_+^{\mathcal{M}}(M_+)$ (which does not intersect K by definition of M_+) and recalling that the coefficients of $\delta_s A[h^s]$ are null outside K , we conclude that $(\delta_s A[h^s]) (\chi_+^a \mathbf{A}) = 0$, hence we can add the term

$$e_A A(\chi_-^r e_A^r (\delta_s A[h^s]) (\chi_+^a \mathbf{A}))$$

to \mathbf{Y} without any problem. In this way we obtain

$$\mathbf{Y} = e_A A(\chi_-^r e_A^r (\delta_s A[h^s]) \mathbf{A}).$$

Now take into account the term $\chi_-^r e_A^a (\delta_s A[h^s]) \mathbf{A}$: the coefficients of $\delta_s A[h^s]$ are supported inside K , hence

$$\text{supp}(e_A^a (\delta_s A[h^s]) \mathbf{A}) \subseteq J_+^{\mathcal{M}}(K),$$

while χ_-^r is supported inside $J_-^{\mathcal{M}}(M_-)$. This entails that $\chi_-^r e_A^a (\delta_s A [h^s]) \mathbf{A} = 0$, therefore we can modify again our expression for \mathbf{Y} subtracting the term

$$e_A A (\chi_-^r e_A^a (\delta_s A [h^s]) \mathbf{A}) = 0.$$

The result is

$$\mathbf{Y} = -e_A A (\chi_-^r e_A (\delta_s A [h^s]) \mathbf{A}).$$

Now we focus our attention on the term $\theta = (\delta_s A [h^s]) \mathbf{A}$. First of all we notice that it is an element of $\Omega_0^1 M$ supported inside K because of the support properties of the coefficients of $\delta_s A [h^s]$. Applying the Leibniz rule in reverse we find the following chain of equalities:

$$\delta \theta = \delta_s (\delta [h^s] A [h^s] \mathbf{A}) - (\delta_s \delta [h^s]) A \mathbf{A} = 0, \quad (3.2.40)$$

where we exploited $A \mathbf{A} = 0$ and $\delta [h^s] \circ A [h^s] = 0$. This proves that θ is also coclosed, hence $e_A \theta$ is a Lorentz solution. In particular we have $A e_A \theta = 0$ and also $\delta e_A \theta = 0$. Then it follows that

$$\begin{aligned} A (\chi_-^r e_A \theta) &= -A (\chi_-^a e_A \theta), \\ \delta (\chi_-^r e_A \theta) &= -\delta (\chi_-^a e_A \theta). \end{aligned}$$

From the last identities, exploiting $\text{supp}(e_A \theta) \subseteq J_-^{\mathcal{M}}(K)$, $\text{supp}(\chi_-^{a/r}) = J_{\pm}^{\mathcal{M}}(\Sigma_-^{a/r})$ for proper smooth spacelike Cauchy surfaces $\Sigma_-^{a/r}$ for \mathcal{M} and Proposition 1.2.18, we deduce that the sections $A(\chi_-^{a/r} e_A \theta)$ and $\delta(\chi_-^{a/r} e_A \theta)$ have compact support. Therefore we can use eq. (3.2.39) for $s = 0$ to obtain the following result:

$$\begin{aligned} \mathbf{Y} &= e_A^a A (\chi_-^a e_A \theta) + e_A^r A (\chi_-^r e_A \theta) \\ &= \chi_-^a e_A \theta - \text{de}_A^a \delta (\chi_+^a e_A \theta) + \chi_-^r e_A \theta - \text{de}_A^r \delta (\chi_+^r e_A \theta) \\ &= e_A \theta - \text{de}_A \delta (\chi_+^a e_A \theta). \end{aligned}$$

The last calculation proves that \mathbf{Y} and $e_A \theta$ are gauge equivalent Lorentz solutions, hence we can consider

$$\mathbf{Z} = e_A \theta = e_A (\delta_s A [h^s]) \mathbf{A}$$

as new representative of the same equivalence class, i.e.

$$\mathbf{Z} \in \left[e_A A [h] \left(\chi_-^{a/r} e_{A[h]} A \left(\chi_+^{a/r} \mathbf{A} \right) \right) \right]_M.$$

With this we conclude

$$\left. \frac{d}{ds} r_{h^s}^{\mathcal{M}} [\mathbf{A}]_M \right|_0 = \left[e_A \left(\left. \frac{d}{ds} A [h^s] \right|_0 \right) \mathbf{A} \right]_M, \quad (3.2.41)$$

where \mathbf{A} is a representative of the class $[\mathbf{A}]_M$. Note that also now the result does not depend on the choice of the particular representative of $[\mathbf{A}]_M$ since, if we consider two representatives in the same equivalence class, they differ by a closed form $\tilde{\mathbf{A}}$, i.e. $d\tilde{\mathbf{A}} = 0$, hence also $A[h^s]\tilde{\mathbf{A}} = 0$ for each s . This fact anyway is trivial since the original expression for $\delta_s r_{h^s}^{\mathcal{M}}[\mathbf{A}]_M$ was independent of the choice of the representative and now we simply looked for a convenient representative in the same equivalence class.

The evaluation of $\delta_s A[h^s]\mathbf{A}$ can be carried on as for the case of the Proca field (as a matter of fact the only difference relies in the absence of the mass term, which is irrelevant for this calculation since it does not depend on s). For convenience we quote here the result:

$$\left. \frac{d}{ds} (A[h^s]\mathbf{A})_k \right|_0 = \left. \frac{d}{ds} h_{ij}^s \right|_0 \nabla^i \mathbf{F}^j_k + \left. \frac{d}{ds} \Gamma[h^s]_{ij}^l \right|_0 g^{ij} \mathbf{F}_{lk} + \left. \frac{d}{ds} \Gamma[h^s]_{ik}^l \right|_0 g^{ij} \mathbf{F}_{jl}, \quad (3.2.42)$$

where \mathbf{F} denotes the field strength associated to \mathbf{A} , i.e. $\mathbf{F} = d\mathbf{A}$ or, in local coordinates,

$$\mathbf{F}_{ij} = \partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i, \quad (3.2.43)$$

which can be expressed in a manifestly covariant manner on both \mathcal{M} and $\mathcal{M}[h^s]$ (for each s) because the terms involving the Christoffel symbols cancel out due to their symmetry (cfr. eq. (1.1.2)):

$$\nabla_i \mathbf{A}_j - \nabla_j \mathbf{A}_i = \mathbf{F}_{ij} = \nabla_i [h^s] \mathbf{A}_j - \nabla_j [h^s] \mathbf{A}_i.$$

Note that the RHS of eq. (3.2.42) is independent of the choice of the representative \mathbf{A} since only the field strength $\mathbf{F} = d\mathbf{A}$ appears.

Properties of the GNS representation induced by a quasi-free Hadamard state for the electromagnetic field

At this point we choose a quasi-free Hadamard state τ on the CCR representation $(\mathcal{V}, \mathcal{V}) = \mathcal{A}(\mathcal{M}, \Lambda^1 M, A)$ describing the electromagnetic field on the globally hyperbolic spacetime \mathcal{M} . With this choice, we introduce the (unique up to unitary equivalence) GNS triple $(\mathcal{H}_\tau^{\mathcal{M}}, \pi_\tau^{\mathcal{M}}, \Omega_\tau^{\mathcal{M}})$ induced by τ and we follow the discussion made in Subsection 3.2.1. In this way we obtain the represented version

$$\mathbf{V}_\tau^{\mathcal{M}} = \pi_\tau^{\mathcal{M}} \circ \mathbf{V} : \mathcal{V} \rightarrow \mathcal{B}(\mathcal{H}_\tau^{\mathcal{M}}) \quad (3.2.44)$$

of the Weyl map \mathbf{V} , where $(\mathcal{V}, \sigma) = \mathcal{B}(\mathcal{M}, \Lambda^1 M, A)$ is the symplectic space provided by the covariant functor describing the classical theory of the electromagnetic field,

and the map

$$\begin{aligned}\Phi_\tau^\mathcal{M} : V &\rightarrow \mathcal{B}(\mathcal{H}_\tau^\mathcal{M}) \\ [\mathbf{A}]_M &\mapsto \Phi_\tau^\mathcal{M}([\mathbf{A}]_M)\end{aligned}$$

which maps each element $[\mathbf{A}]_M$ of V to a selfadjoint operator $\Phi_\tau^\mathcal{M}([\mathbf{A}]_M)$ on $\mathcal{H}_\tau^\mathcal{M}$. Moreover for each $[\mathbf{A}]_M \in V$ it holds that

$$e^{i\Phi_\tau^\mathcal{M}([\mathbf{A}]_M)} = V_\tau^\mathcal{M}([\mathbf{A}]_M).$$

Together with the map $\Phi_\tau^\mathcal{M}$, we have the smeared fields (this is a consequence of the choice of a Hadamard state):

$$\begin{aligned}\Psi_\tau^\mathcal{M} : \Omega_{0,\delta}^1 M &\rightarrow \mathcal{B}(\mathcal{H}_\tau^\mathcal{M}) \\ \theta &\mapsto -i \frac{d}{dt} V_\tau^\mathcal{M}(t[e_A\theta]_M) \Big|_0.\end{aligned}$$

Note that here appears a slight difference with respect to the previous cases, namely that the test section we consider are coclosed. As for the general case, it holds that

$$\Psi_\tau^\mathcal{M}(\theta) = \Phi_\tau^\mathcal{M}([e_A\theta]_M) \quad (3.2.45)$$

for each $\theta \in \Omega_{0,\delta}^1 M$ and we recognize $\Psi_\tau^\mathcal{M}$ to be linear.

Also in this case the choice of a quasi-free Hadamard state τ assures that Assumption 3.1.5 holds, i.e. we are able to find a dense subspace $\mathcal{V}_\tau^\mathcal{M}$ of $\mathcal{H}_\tau^\mathcal{M}$ and a dense sub- \ast -algebra $\mathcal{B}_\tau^\mathcal{M}$ of $\mathcal{A}(\mathcal{M}, \Lambda^1 M, A)$ such that the functional derivative of the RCE with respect to the spacetime metric can be defined. In particular $\mathcal{V}_\tau^\mathcal{M}$ is constituted by all the vectors of the form $L\Omega_\tau^\mathcal{M}$, where L is an arbitrary polynomial in $V_\tau^\mathcal{M}([\mathbf{A}]_M)$ and $\Psi_\tau^\mathcal{M}(\theta)$ for arbitrary $[\mathbf{A}]_M \in V$ and $\theta \in \Omega_{0,\delta}^1 M$.

Again we have an equation similar to eq. (3.2.3): for each $\xi \in \mathcal{V}_\tau^\mathcal{M}$, each $[\mathbf{A}]_M \in V$, each compact subset K of M and each smooth 1-parameter family $(-1, 1) \rightarrow GHP(\mathcal{M}, K)$, $s \mapsto h^s$ such that $h^0 = 0$, it holds that

$$\frac{d}{ds} \left\langle \xi, V_\tau^\mathcal{M}(r_{h^s}^\mathcal{M}[\mathbf{A}]_M) \xi \right\rangle_\tau \Big|_0 = \frac{i}{2} \left\langle \xi, \left\{ \Phi_\tau^\mathcal{M} \left(\frac{d}{ds} (r_{h^s}^\mathcal{M}[\mathbf{A}]_M) \Big|_0 \right), V_\tau^\mathcal{M}([\mathbf{A}]_M) \right\} \xi \right\rangle_\tau^\mathcal{M}. \quad (3.2.46)$$

Moreover one can show that for each $\eta, \xi \in \mathcal{V}_\tau^\mathcal{M}$ there exists a smooth section, that we denote with

$$\begin{aligned}M &\rightarrow T_\mathbb{C}M \\ p &\mapsto \langle \eta, \Psi_\tau^\mathcal{M}(p) \xi \rangle_\tau^\mathcal{M},\end{aligned} \quad (3.2.47)$$

where $T_\mathbb{C}M$ stands for the complex vector bundle obtained via the tensor product

of each fiber of TM with \mathbb{C} and $\langle \cdot, \cdot \rangle_\tau^{\mathcal{M}}$ denotes the scalar product of the Hilbert space $\mathcal{H}_\tau^{\mathcal{M}}$, such that

$$\langle \eta, \Psi_\tau^{\mathcal{M}}(\theta) \xi \rangle_\tau^{\mathcal{M}} = \int_M (\theta(p)) \left(\langle \eta, \Psi_\tau^{\mathcal{M}}(p) \xi \rangle_\tau^{\mathcal{M}} \right) d\mu_g \quad (3.2.48)$$

for each $\theta \in \Omega_{0,\delta}^1 M$, where $d\mu_g$ is the standard volume form on \mathcal{M} and the dual pairing between T^*M and TM has been taken into account (note that one may indeed write the integrand using the abstract index notation putting a contravariant index on the new section and a covariant index on the test function).

Remark 3.2.8. We meet here the first consequence of the restriction of the set of test sections to $\Omega_{0,\delta}^1 M$, namely that $\langle \eta, \Psi_\tau^{\mathcal{M}}(p) \xi \rangle_\tau^{\mathcal{M}}$ fails to be unique: as a matter of fact each section that differs from this one by an exact 1-form (with raised indices) will do the work perfectly well. On the contrary, if there are two sections satisfying eq. (3.2.48), we deduce that they differ by a closed one form (with raised indices).

It seems that we fail to have a characterization of $\langle \eta, \Psi_\tau^{\mathcal{M}}(p) \xi \rangle_\tau^{\mathcal{M}}$ in terms of a class of gauge equivalent sections since we are not sure that we obtain a section satisfying eq. (3.2.48) if we add a closed form (with raised indices) to a section that satisfies eq. (3.2.48). However we required that the first de Rham cohomology group of the manifolds over which we discuss the electromagnetic field is trivial (see Definition 2.3.15), hence each closed 1-form is also exact. This hypothesis restores the usual notion of gauge equivalence also for $\langle \eta, \Psi_\tau^{\mathcal{M}}(p) \xi \rangle_\tau^{\mathcal{M}}$ because now exactness and closure of 1-forms coincide.

Indeed from a physical point of view we expected to find a counterpart of the gauge equivalence for the electromagnetic field at the quantum level.

As we will see, the lack of uniqueness for $\langle \eta, \Psi_\tau^{\mathcal{M}}(p) \xi \rangle_\tau^{\mathcal{M}}$ will not affect our calculation. For the moment we regard the section in eq. (3.2.47) as (the matrix element of) one of the gauge equivalent unsmeared electromagnetic fields induced by the quasi-free Hadamard state τ on the globally hyperbolic spacetime \mathcal{M} .

Quantized stress-energy tensor for the electromagnetic field

Our final theorem requires that we know how to express the quantized stress-energy tensor for the electromagnetic field. As always, we use as a starting point the equation governing the classical dynamics of the field to obtain a natural expression for the action associated to the field itself. After that, we determine the classical stress-energy tensor for the electromagnetic field evaluating the functional derivative of the action with respect to the spacetime metric and we try to determine the corresponding quantum observable via the point-splitting procedure. At the classical level the situation is identical to the Proca field provided that we set $m = 0$ (this is due to the fact that the linear differential operator governing the classical dynamics

of the electromagnetic field, i.e. δd , is nothing but the one for the Proca field with $m = 0$). For the action on the globally hyperbolic spacetime \mathcal{M} we obtain the following expression:

$$S_{\mathcal{M}} = \frac{1}{2} (A, AA)_{g,1} = \frac{1}{2} (dA, dA)_{g,2} = \frac{1}{2} \int_M (dA \wedge *dA).$$

Evaluating the functional derivative of $S_{\mathcal{M}}$ with respect to the metric, we find the classical stress-energy tensor for the electromagnetic field (we express it in local coordinates):

$$\begin{aligned} T_{ij}^{\mathcal{M}}(p) &= \frac{2}{\sqrt{|\det g_h(p)|}} \frac{\delta S_{\mathcal{M}[h]}}{\delta g_h^{ij}(p)} \Big|_0 \\ &= g^{bd}(p) F_{ib}(p) F_{jd}(p) - \frac{1}{4} g_{ij}(p) g^{ac}(p) g^{bd}(p) F_{ab}(p) F_{cd}(p), \end{aligned}$$

where F is defined according to eq. (3.2.43). The choice of a Hadamard state allows us to promote $T_{ij}^{\mathcal{M}}$ to the renormalized quantum stress-energy tensor $\mathcal{T}_{\tau ij}^{\mathcal{M}}$ simply via point-splitting (refer to [Wal94, eq. 4.6.5, p. 88]): For each $\eta, \xi \in \mathcal{V}_{\tau}^{\mathcal{M}}$, we choose two “near” points p and q in M and a curve γ connecting them and, parallel transporting along the curve γ , we write

$$\begin{aligned} \langle \eta, \mathcal{T}_{\tau}^{\mathcal{M} ij}(p, q) \xi \rangle_{\tau}^{\mathcal{M}} &= g_{bf}(p) Y_{\gamma b}^f \langle \eta, \Pi_{\tau}^{\mathcal{M} ib}(p) \Pi_{\tau}^{\mathcal{M} jd}(q) \xi \rangle_{\tau}^{\mathcal{M}} \\ &\quad - \frac{1}{4} g^{ik}(p) Y_{\gamma k}^j g_{ae}(p) Y_{\gamma c}^e g_{bf}(p) Y_{\gamma d}^f \langle \eta, \Pi_{\tau}^{\mathcal{M} ab}(p) \Pi_{\tau}^{\mathcal{M} cd}(q) \xi \rangle_{\tau}^{\mathcal{M}}, \end{aligned} \quad (3.2.49)$$

where we considered

$$\Pi_{\tau}^{\mathcal{M} ij}(p) = \nabla^i \Psi_{\tau}^{\mathcal{M} j}(p) - \nabla^j \Psi_{\tau}^{\mathcal{M} i}(p) \quad (3.2.50)$$

to shorten the expression.

If we lower all the indices in eq. (3.2.50), we realize that $\Pi_{\tau}^{\mathcal{M}}$ is nothing but the exterior derivative of the (non unique) unsmeared field. This fact entails that $\Pi_{\tau}^{\mathcal{M}}$ does not depend on the particular choice of the unsmeared field because, even if we add a closed 1-form, then the exterior derivative set this contribution to zero. A direct consequence of this fact is the independence of eq. (3.2.49) on the choice of an unsmeared field because only $\Pi_{\tau}^{\mathcal{M} ij}(p)$ appears on the RHS. Indeed all these observations must be intended in the sense of matrix elements (which are the only ones that we defined so far).

As we observed when we dealt with the Klein-Gordon field in $\mathcal{T}_{\tau ij}^{\mathcal{M}}$ there is no dependence upon the choice of the curve γ along which we parallel transport provided that the points p and q are sufficiently near so that there exists only one geodesic connecting them and we consider such geodesic as γ . Indeed we can take p and q in a sufficiently small neighborhood since our aim is to take the limit $q \rightarrow p$ along γ once

that we have found an expression that does not present divergences in this limit. We also stress that the quantized stress-energy tensor obtained via point-splitting differs by a multiple of the identity operator from the quantized stress-energy tensor provided by the regularization procedure with τ as reference state. Anyway we are only interested in the commutator of the stress-energy tensor with some represented Weyl generator, hence such difference is irrelevant in our computations.

As we said few lines above, our upcoming theorem will involve the stress-energy tensor only in a commutator with some represented Weyl generator $V_\tau^{\mathcal{M}}([A]_M)$ for $[A]_M \in V$, where $(V, \sigma) = \mathcal{B}(\mathcal{M}, \Lambda^1 M, A)$ for a fixed object $(\mathcal{M}, \Lambda^1 M, A)$ in \mathbf{ghs}^{EM} . Reading eq. (3.2.49) we realize that it would be useful to evaluate the matrix elements of the commutator of $\Pi_\tau^{\mathcal{M}}(p) \Pi_\tau^{\mathcal{M}}(q)$ with an arbitrary represented Weyl generator. To this end we recall eq. (3.2.2) and we evaluate its LHS and its RHS fixing $\eta, \xi \in \mathcal{V}_\tau^{\mathcal{M}}, \theta, \theta' \in \Omega_{0,\delta}^1 M$ and $[A]_M \in V$ using eq. (3.2.48) twice (all the equations are written using the abstract index notation):

$$\begin{aligned} \iint_M \langle \eta, [\Psi_\tau^{\mathcal{M}i}(p) \Psi_\tau^{\mathcal{M}j}(q), V_\tau^{\mathcal{M}}([A]_M)] \xi \rangle_\tau^{\mathcal{M}} \theta_i(p) \theta'_j(q) d\mu_g(p) d\mu_g(q) \\ = \langle \eta, [\Psi_\tau^{\mathcal{M}}(\theta) \Psi_\tau^{\mathcal{M}}(\theta'), V_\tau^{\mathcal{M}}([A]_M)] \xi \rangle_\tau^{\mathcal{M}}. \end{aligned}$$

Indeed the matrix element inside the integral on the LHS of the last equation is not unique because of the gauge invariance. Now we exploit also the definition of the symplectic form σ (cfr. Lemma 2.3.14):

$$\begin{aligned} -\sigma([e_A \theta]_M, [A]_M) \langle \eta, V_\tau^{\mathcal{M}}([A]_M) \Psi_\tau^{\mathcal{M}}(\theta') \xi \rangle_\tau^{\mathcal{M}} \\ = \iint_M A_k(p) g^{ki}(p) \theta_i(p) \theta'_j(q) \langle \eta, V_\tau^{\mathcal{M}}([A]_M) \Psi_\tau^{\mathcal{M}j}(q) \xi \rangle_\tau^{\mathcal{M}} d\mu_g(p) d\mu_g(q), \\ -\sigma([e_A \theta']_M, [A]_M) \langle \eta, \Psi_\tau^{\mathcal{M}}(\theta) V_\tau^{\mathcal{M}}([A]_M) \xi \rangle_\tau^{\mathcal{M}} \\ = \iint_M A_k(q) g^{kj}(q) \theta'_j(q) \theta_i(p) \langle \eta, \Psi_\tau^{\mathcal{M}i}(p) V_\tau^{\mathcal{M}}([A]_M) \xi \rangle_\tau^{\mathcal{M}} d\mu_g(p) d\mu_g(q), \end{aligned}$$

where A is some representative of the class $[A]_M$. These integrals present the terms

$$\begin{aligned} A^i(p) \langle \eta, V_\tau^{\mathcal{M}}([A]_M) \Psi_\tau^{\mathcal{M}j}(q) \xi \rangle_\tau^{\mathcal{M}}, \\ \langle \eta, V_\tau^{\mathcal{M}}([A]_M) \Psi_\tau^{\mathcal{M}i}(qp) \xi \rangle_\tau^{\mathcal{M}} A^j(q) \end{aligned}$$

which are not uniquely defined exactly as seen above. From eq. (3.2.2) and the

freedom in the choice of θ and θ' we deduce that

$$\begin{aligned} & \langle \eta, [\Psi_{\tau}^{\mathcal{M}i}(p) \Psi_{\tau}^{\mathcal{M}j}(q), V_{\tau}^{\mathcal{M}}([A]_M)] \xi \rangle_{\tau}^{\mathcal{M}} \\ & \sim A^i(p) \langle \eta, V_{\tau}^{\mathcal{M}}([A]_M) \Psi_{\tau}^{\mathcal{M}j}(q) \xi \rangle_{\tau}^{\mathcal{M}} + A^j(q) \langle \eta, \Psi_{\tau}^{\mathcal{M}i}(p) V_{\tau}^{\mathcal{M}}([A]_M) \xi \rangle_{\tau}^{\mathcal{M}} \end{aligned} \quad (3.2.51)$$

for each choice of A in the class $[A]_M$, where \sim means gauge equivalence. All the terms that we could add without affecting the relation \sim cancel out once that we evaluate

$$\langle \eta, [\Pi_{\tau}^{\mathcal{M}ij}(p) \Pi_{\tau}^{\mathcal{M}kl}(q), V_{\tau}^{\mathcal{M}}(\Theta)] \xi \rangle_{\tau}^{\mathcal{M}},$$

hence from eq. (3.2.51) we deduce that

$$\begin{aligned} & \langle \eta, [\Pi_{\tau}^{\mathcal{M}ij}(p) \Pi_{\tau}^{\mathcal{M}kl}(q), V_{\tau}^{\mathcal{M}}(\Theta)] \xi \rangle_{\tau}^{\mathcal{M}} \\ & = F^{ij}(p) \langle \eta, V_{\tau}^{\mathcal{M}}(\Theta) \Pi_{\tau}^{\mathcal{M}kl}(q) \xi \rangle_{\tau}^{\mathcal{M}} + F^{kl}(q) \langle \eta, \Pi_{\tau}^{\mathcal{M},ij}(p) V_{\tau}^{\mathcal{M}}(\Theta) \xi \rangle_{\tau}^{\mathcal{M}}. \end{aligned} \quad (3.2.52)$$

This is the relation that we will use in the proof of the next theorem.

Main theorem

We are ready to state and prove the main theorem of this subsection. Such theorem extends to the case of the electromagnetic field the results of compatibility between the action of the functional derivative of the relative Cauchy evolution and the stress-energy tensor, which are already known to hold for the Klein-Gordon field and the Proca field (refer to Subsection 3.2.2 and to Subsection 3.2.3).

Theorem 3.2.9. *Let $\mathcal{A} : \mathfrak{ghs}^{EM} \rightarrow \mathfrak{alg}$ be the locally covariant quantum field theory for the electromagnetic field built in Subsection 2.3.3 and let $(\mathcal{M}, \Lambda^1 M, A)$ be an object of the category \mathfrak{ghs}^{EM} (see Definition 2.3.15). Consider a quasi-free Hadamard state τ on the CCR representation $(\mathcal{V}, \mathcal{V}) = \mathcal{A}(\mathcal{M}, \Lambda^1 M, A)$ and denote the GNS triple induced by τ with $(\mathcal{H}_{\tau}^{\mathcal{M}}, \pi_{\tau}^{\mathcal{M}}, \Omega_{\tau}^{\mathcal{M}})$. We denote with $V_{\tau}^{\mathcal{M}}$ the represented counterpart of the Weyl map V (cfr. eq. (3.2.44)) and with $\mathcal{T}_{\tau}^{\mathcal{M}}$ the quantum stress-energy tensor for the electromagnetic field on \mathcal{M} obtained via point-splitting in the representation induced by the state τ (cfr. eq. (3.2.49)). Then there exists a dense subspace $\mathcal{V}_{\tau}^{\mathcal{M}}$ of $\mathcal{H}_{\tau}^{\mathcal{M}}$ such that*

$$\frac{\delta}{\delta h} \pi_{\tau}^{\mathcal{M}}(R_h^{\mathcal{M}}(V([A]_M))) = -\frac{i}{2} [\mathcal{T}_{\tau}^{\mathcal{M}}, V_{\tau}^{\mathcal{M}}([A]_M)] \quad \forall [A]_M \in V$$

in the sense of quadratic forms on $\mathcal{V}_{\tau}^{\mathcal{M}}$.

Proof. A dense subspace $\mathcal{V}_{\tau}^{\mathcal{M}}$ of $\mathcal{H}_{\tau}^{\mathcal{M}}$ exists by virtue of the choice of a quasi-free Hadamard state τ (see few lines before eq. (3.2.46)).

We fix $\xi \in \mathcal{V}_{\tau}^{\mathcal{M}}$, $[A]_M \in V$, a compact subset K of M and 1-parameter family $(-1, 1) \rightarrow GHP(\mathcal{M}, K)$, $s \mapsto h^s$ such that $h^0 = 0$. We repeat the first part of the

proof of Theorem 3.2.4 using eq. (3.2.36) and eq. (3.2.46) in place of eq. (3.2.4) and respectively eq. (3.2.12). In this way we find the following reformulation of the thesis:

$$\begin{aligned} \langle \xi, \{ \Phi_\tau^{\mathcal{M}} (\delta_s r_{h^s}^{\mathcal{M}} [\mathbf{A}]_M), V_\tau^{\mathcal{M}} ([\mathbf{A}]_M) \} \xi \rangle_\tau^{\mathcal{M}} \\ = - \int_M (\delta_s h^s) \left(\langle \xi, [\mathcal{T}_\tau^{\mathcal{M}}, V_\tau^{\mathcal{M}} ([\mathbf{A}]_M)] \xi \rangle_\tau^{\mathcal{M}} \right) d\mu_g, \end{aligned}$$

where the dual pairing between $T^*M \otimes_s T^*M$ and $TM \otimes_s TM$ is taken into account in the integrand appearing on the RHS. Now we exploit eq. (3.2.41) choosing a representative \mathbf{A} of the fixed class $[\mathbf{A}]_M$:

$$\begin{aligned} \langle \xi, \{ \Phi_\tau^{\mathcal{M}} ([e_A \delta_s A [h^s] \mathbf{A}]_M), V_\tau^{\mathcal{M}} ([\mathbf{A}]_M) \} \xi \rangle_\tau^{\mathcal{M}} \\ = - \int_M (\delta_s h^s) \left(\langle \xi, [\mathcal{T}_\tau^{\mathcal{M}}, V_\tau^{\mathcal{M}} ([\mathbf{A}]_M)] \xi \rangle_\tau^{\mathcal{M}} \right) d\mu_g. \end{aligned}$$

Note that the result does not depend on the particular choice of \mathbf{A} in the class $[\mathbf{A}]_M$ because the same is true for eq. (3.2.41). We can apply also eq. (3.2.45) since eq. (3.2.40) shows that $\delta_s A [h^s] \mathbf{A}$ is coclosed whatever choice of \mathbf{A} we make:

$$\begin{aligned} \overbrace{\langle \xi, \{ \Psi_\tau^{\mathcal{M}} (\delta_s A [h^s] \mathbf{A}), V_\tau^{\mathcal{M}} ([\mathbf{A}]_M) \} \xi \rangle_\tau^{\mathcal{M}}}^{\text{L}} \\ = - \underbrace{\int_M (\delta_s h^s) \left(\langle \xi, [\mathcal{T}_\tau^{\mathcal{M}}, V_\tau^{\mathcal{M}} ([\mathbf{A}]_M)] \xi \rangle_\tau^{\mathcal{M}} \right) d\mu_g}_{\text{R}}. \end{aligned}$$

Now we work with the LHS of the last equation (denoted by L) and the RHS (denoted by R) separately. Starting from L, we exploit the relation between smeared and unsmeared fields, eq. (3.2.48):

$$\text{L} = \int_M ((\delta_s A [h^s] \mathbf{A})(p)) \left(\langle \xi, \{ \Psi_\tau^{\mathcal{M}} (p), V_\tau^{\mathcal{M}} ([\mathbf{A}]_M) \} \xi \rangle_\tau^{\mathcal{M}} \right) d\mu_g,$$

where we consider the dual pairing between T^*M and TM . Indeed the integrand on the right is not uniquely determined because of gauge equivalence. Anyway every admissible choice of this section will give the same value for L. In order to find an expression for L in local coordinates, we perform the usual construction which provides a finite family $\{(U_\alpha, V_\alpha, \phi_\alpha)\}$ obtained choosing all the elements of a locally finite covering of M constituted by oriented coordinate neighborhoods that intersect the fixed compact subset K of M (which includes the support of the coefficients appearing in $\delta_s A [h^s]$). As usual the choice of the oriented coordinate neighborhoods is made in such a way that $|\det g| = 1$ so that $d\mu_g$ reduces to the standard volume

form dV on \mathbb{R}^4 on each coordinate neighborhood. At the same time we take only the corresponding members $\{\chi_\alpha\}$ in the partition of unity subordinate to the original locally finite covering. In this way we obtain the expression of L in local coordinates:

$$L = \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \langle \xi, \{ \Psi_{\tau}^{\mathcal{M}i}(x), V_{\tau}^{\mathcal{M}}([A]_M) \} \xi \rangle_{\tau}^{\mathcal{M}} (\delta_s A[h^s] A)_i(x) dV,$$

where all the sections that appear inside the integral are now written in local coordinates, namely $\delta_s A[h^s] A$ inside the integral over V_{α} denotes the push-forward through ϕ_{α} of the original $(\delta_s A[h^s] A)$ restricted to U_{α} and similarly for the other sections inside the integral. It is convenient to define

$$\begin{aligned} \zeta : M &\rightarrow T_{\mathbb{C}}M \\ p &\mapsto \langle \xi, \{ \Psi_{\tau}^{\mathcal{M}}(x), V_{\tau}^{\mathcal{M}}([A]_M) \} \xi \rangle_{\tau}^{\mathcal{M}} \end{aligned}$$

in order to simplify our notation. Indeed ζ is not uniquely determined so that we fix some proper ζ and we show that everything works whatever choice of ζ we make. The next two steps are identical to the corresponding ones in the proof of Theorem 3.2.6, provided that we use F defined in eq. (3.2.41) in place of Π : In first place we use eq. (3.2.42) and in second place we partially integrate. We get the following result:

$$\begin{aligned} L = & \underbrace{- \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} (\nabla^i \zeta^k) F_{jk}^j \delta_s h_{ij}^s dV}_{L_1} + \underbrace{\sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^k F_{lk} \delta_s \Gamma [h^s]_{ij}^l g^{ij} dV}_{L_2} \\ & + \underbrace{\sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^k F_{jl} g^{ij} \delta_s \Gamma [h^s]_{ik}^l dV}_{L_3} - \underbrace{\sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta_k F^{jk} \nabla^i \delta_s h_{ij}^s dV}_{L_4}. \end{aligned}$$

Now we focus on R and we express it using the local coordinates $\{(U_{\alpha}, V_{\alpha}, \phi_{\alpha})\}$:

$$R = - \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha}(x) (\delta_s h_{ij}^s)(x) \langle \xi, [\mathcal{T}_{\tau}^{\mathcal{M}ij}(x), V_{\tau}^{\mathcal{M}}([A]_M)] \xi \rangle_{\tau}^{\mathcal{M}} dV.$$

Now recall the expression of the quantized stress-energy tensor, eq. (3.2.49), and the commutation relation found in eq. (3.2.52). Exploiting these results, evaluate $\langle \xi, [\mathcal{T}_{\tau}^{\mathcal{M}ij}(p, q), V_{\tau}^{\mathcal{M}}([A]_M)] \xi \rangle_{\tau}^{\mathcal{M}}$. That done, take the coincidence limit as required by the point-splitting procedure (note that no divergence arises) and insert the result into the last equation. After that, use the symmetry of $\delta_s h^s$ and g to simplify the expression (matrix elements of anticommutators should appear). All these operations produce the following result (to shorten the expression we denote

$\langle \xi, \{ \Pi_\tau^\mathcal{M}(x), V_\tau^\mathcal{M}([A]_M) \} \xi \rangle_\tau^\mathcal{M}$ with Ξ):

$$R = - \underbrace{\sum_\alpha \int_{V_\alpha} \chi_\alpha \Xi^{ib} F_b^j \delta_s h_{ij}^s dV}_{R_1} + \underbrace{\frac{1}{4} \sum_\alpha \int_{V_\alpha} \chi_\alpha F_{ab} \Xi^{ab} \delta_s h_{ij}^s g^{ij} dV}_{R_2}.$$

Eq. (3.2.50) and the subsequent remarks entail that

$$\begin{aligned} \Xi^{ij} &= \langle \xi, \{ \Pi_\tau^\mathcal{M ij}, V_\tau^\mathcal{M}([A]_M) \} \xi \rangle_\tau^\mathcal{M} \\ &= \nabla^i \langle \xi, \{ \Psi_\tau^\mathcal{M j}(x), V_\tau^\mathcal{M}([A]_M) \} \xi \rangle_\tau^\mathcal{M} - \nabla^j \langle \xi, \{ \Psi_\tau^\mathcal{M i}(x), V_\tau^\mathcal{M}([A]_M) \} \xi \rangle_\tau^\mathcal{M} \end{aligned}$$

does not depend on the particular choice of the non unique matrix element of the unsmeared field. In particular we can use the section ζ previously fixed:

$$\Xi^{ij} = \nabla^i \zeta^j - \nabla^j \zeta^i.$$

We denote the first part of R with R_1 and the second with R_2 . In first place we evaluate R_1 by partial integration (we omit the term including derivatives of χ_α since as always they give null contribution):

$$\begin{aligned} R_1 &= \overbrace{- \sum_\alpha \int_{V_\alpha} \chi_\alpha (\nabla^i \zeta^k) F_k^j \delta_s h_{ij}^s dV}^{=X} + \sum_\alpha \int_{V_\alpha} \chi_\alpha (\nabla^b \zeta^i) F_b^j \delta_s h_{ij}^s dV \\ &= X - \underbrace{\sum_\alpha \int_{V_\alpha} \chi_\alpha \zeta^k F^{jb} \nabla_b \delta_s h_{kj}^s dV}_{R_3} + \sum_\alpha \int_{V_\alpha} \chi_\alpha \zeta^i g^{jk} \underbrace{(\nabla^b F_{bk})}_{=0} \delta_s h_{ij}^s dV, \end{aligned}$$

where we recognized the term X already present in L , we exploited the fact that

$$\nabla^i F_{ij} = (\delta dA)_j = 0$$

because every representative A of the class $[A]_M$ satisfies $AA = \delta dA = 0$ and we denoted with R_3 the remaining term. We have the following result:

$$R_1 = X + R_3.$$

In second place we evaluate R_2 proceeding with the same approach. First of all we notice that we can exploit the antisymmetry of F to simplify a little bit the first integral. Then we partially integrate such term with the purpose of finding another integrand that explicitly exhibits the structure of the field equation, i.e. a term $\nabla^i F_{ij}$, so that we can get rid of it too (again we omit the null term containing

derivatives of χ_α):

$$\begin{aligned}
 R_2 &= \frac{1}{2} \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} F_{ab} (\nabla^a \zeta^b) \delta_s h_{ij}^s g^{ij} dV \\
 &= -\frac{1}{2} \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^b F_{ab} \nabla^a \delta_s h_{ij}^s g^{ij} dV - \frac{1}{2} \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^b \underbrace{(\nabla^a F_{ab})}_{=0} \delta_s h_{ij}^s g^{ij} dV \\
 &= -\frac{1}{2} \sum_{\alpha} \int_{V_{\alpha}} \chi_{\alpha} \zeta^b F_{ab} \nabla^a \delta_s h_{ij}^s g^{ij} dV.
 \end{aligned}$$

Therefore, renaming some summation indices, we obtain the following result:

$$R = X + R_2 + R_3.$$

At this stage our thesis $L = R$ is reduced to the following identity:

$$L_2 + L_3 + L_4 = R_2 + R_3.$$

One immediately realizes that eq. (3.2.34) (with F in place of Π) and eq. (3.2.35) imply our last equation: to recognize this fact proceed as we did after eq. (3.2.35) in the case of the Proca field.

Eq. (3.2.35) is a purely geometrical identity, hence holds also in this case without any further comment. On the contrary eq. (3.2.34) involves an object strictly connected with the dynamics of the Proca field (namely Π), however the proof of this identity relies only on the antisymmetry of such object, a property that indeed holds also for F , hence a similar identity holds for F in place of Π . These observations entail that we have $L = R$ whatever choice of ζ we make. This completes the proof. \square

Conclusions

In Chapter 1 we introduced almost all the mathematical tools required for the entire thesis. Particular attention was devoted to geometrical tools in the context of vector bundles, which constitute the mathematical setting of the whole discussion, together with globally hyperbolic spacetimes. We also recalled some results about normally hyperbolic equations on globally hyperbolic spacetimes. After that we turned our attention to the algebraic tools, namely algebras and states, that are needed to discuss the algebraic approach to quantum field theory. We focused mainly on particular C^* -algebras, namely Weyl systems and CCR representation, which are well suited for the quantization of bosonic fields. To conclude some definitions from category theory were presented, the language of category theory being suitable for a number of notions presented in the thesis.

After the required mathematical preliminaries, the main subject of the thesis was tackled in Chapter 2 with the introduction of the *generally covariant locality principle* (GCLP), originally formulated in [BFV03]. To do this, in first place we analyzed in detail the structure of the category of globally hyperbolic spacetimes and the structure of the category of unital C^* -algebras, taking advantage of the remarks made in Chapter 1. In second place we stated the GCLP giving the definition of *locally covariant quantum field theory* (LCQFT). We devoted particular attention to the physical interpretation of the GCLP, essentially borrowing the interpretation of the Haag-Kastler axioms (refer to [HK64]). We also showed in full detail that it is possible to completely recover the Haag-Kastler axioms starting from the assignment of a locally covariant quantum field theory fulfilling both the causality condition and the time slice axiom. In third place we showed how to realize a LCQFT starting from a normally hyperbolic equation over a globally hyperbolic spacetime. This was done in two steps. The first one consisted in the realization of a covariant functor describing the classical theory of the field whose dynamics is ruled by the assigned normally hyperbolic equation, while the second was realized quantizing such classical field theory via composition with another covariant functor that embodies the quantization procedure. Great care was devoted to study in full detail the properties of the starting category for the classical field functor, which is a sort of enriched category of globally hyperbolic spacetimes. We concluded Chapter 2 with the construction of LCQFTs for the Klein-Gordon field, the Proca field and the electromagnetic field.

While the Klein-Gordon case is nothing more than a specialization of the general procedure, the other two cases required more attention as a consequence of the lack of a normally hyperbolic equation governing their classical dynamics. The case of the electromagnetic field proved to be the most involved. To simplify the situation, we restricted to those field strengths that could be described in terms of a vector potential. Therefore, in place of the Maxwell equations, we considered the resulting equation for the vector potential and we kept into account the effects of gauge equivalence.

Chapter 3 was devoted to the main argument of the thesis, namely the *relative Cauchy evolution (RCE)*. In fact our original purpose was to show that a relation between the RCE and the stress-energy tensor similar to the one proved in [BFV03] for the Klein-Gordon field holds also for the Proca field and the electromagnetic field. In first place we defined in a general context the RCE and its functional derivative with respect to the spacetime metric. We proved that the functional derivative, which is symmetric by construction, is also divergence free, thus finding a hint for a possible strict relation with the stress-energy tensor. After that we returned to the examples discussed at the end of Chapter 2. In first place we proved the relation between the functional derivative of the RCE and the stress-energy tensor originally showed in [BFV03] for the case of the Klein-Gordon field. In second place we tried to extend this result to the Proca field and the electromagnetic field. While the case of the Proca field proved to be almost straightforward (the main difference can be ascribed to the fact that the Proca field is a 1-form, while the Klein-Gordon field is a 0-form), the electromagnetic field presents some additional complications. Anyway we were able to circumvent these obstructions exploiting the gauge equivalence. In this way our purpose was achieved, namely we showed that the relation between the functional derivative of the RCE and stress-energy tensor, which was already known to hold for the Klein-Gordon field, holds in an identical form in the cases of the Proca and the electromagnetic fields too.

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